

# A Universal Behavior of Half BPS Probes in the Superstar Ensemble

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**ABSTRACT:** In this paper we probe the typical states of the superstar ensemble of [1] using half-BPS states of type-IIB string theory on  $\text{AdS}_5 \times \text{S}^5$ . We find a very simple universal result that has the structure  $\log \langle \langle \psi \psi \rangle \rangle_{\mathcal{O}} \approx \alpha h \log N$ , where  $h$  is the conformal weight of the probe  $\psi$  and  $\alpha$  is a constant that depends mainly of the shape of the probe  $\psi$ .

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## 1. Introduction

The thermodynamical nature of black holes was revealed by studying the response of black holes to both perturbations in their defining parameters (mass, angular momentum, charge) [2], as well as their response to the presence of quantum fields in the bulk of their geometry [3, 4]. This unexpected nature of black holes led to the famous black hole puzzles<sup>1</sup>: The black hole entropy puzzle, and the information loss puzzle. A lot of effort was spent in the the last four decades or so to solve these puzzles. Among the ideas that emerged during this investigation is the fuzzball proposal, first advanced by Mathur and collaborators [5, 6, 7]. By now there are many nice reviews on the subject, see for example [7, 8, 9, 10, 11, 12, 13]. The idea of the fuzzball proposal is that the black hole geometry is an effective description of an underlying exponentially large system of microstates. It is widely believed that some of these microstates manifest themselves as smooth geometries on the gravity side. However, depending on the kind of black holes under consideration, not all of them can have a gravity description [14]. Another point of view that was advanced in [15], is that the black hole geometry is the effective description of a set of microstates that do not have a smooth geometry description. The situation is far from being conclusive and we will not be concerned with these issues in the present paper. For a further discussion on these issues see e.g. [16, 17].

One of the papers that went beyond comparing the macroscopic and microscopic entropies in checking the fuzzball proposal is [1] (see also [18] for a different approach). They studied a specific ensemble of heavy half-BPS states of type-IIB string theory on asymptotic  $\text{AdS}_5 \times S^5$ , called the superstar ensemble, and constructed its effective dual geometry (see [19] for the application of similar ideas to the case of the D1-D5 system). It turned out that this geometry is the same as the one of the superstar of [20]. This led them to conjecture that the superstar is an effective description of the superstar ensemble in line with the fuzzball proposal. They supplemented this claim with further checks using some correlation functions. We initiate in this paper a further check of this proposal by studying the effect of light half-BPS probes on the heavy states of the superstar ensemble. We find that at leading order, the final answer is universal and does not depend on the details of the typical states of the superstar ensemble.

This paper is organized as follows. In the second section we quickly review the superstar ensemble and discuss some important properties of its typical states that will be useful later

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<sup>1</sup>Another puzzle, the singularity puzzle, is usually added to the list. This puzzle questions the validity of general relativity near the singularity and has nothing to do with the thermodynamical nature of black holes. That is why we did not include it in our list.

on. This material is not new and can be found in [1]. After that, we introduce the different classes of probes that we will be dealing with in this paper and discuss some of their properties. In the third section, we introduce the two point function which is the main quantity we will be evaluating in this paper. Essentially, half-BPS states of type-IIB string theory on asymptotic  $\text{AdS}_5 \times \text{S}^5$  can be described using Young diagrams (YDs) which are treated as irreducible representations of the unitary group<sup>2</sup>  $\text{U}(N)$ , where  $N$  is the flux of the background geometry [23, 24]. To evaluate the two point function we end up facing the problem of decomposing the tensor product of two  $\text{U}(N)$  representations into irreducible ones. This is the topic of the fourth section. Although it is impossible to completely carry out this decomposition, we manage to extract enough information about it to be able to evaluate the leading order of the two point function. In the fifth section, we calculate the leading order term of the log of the two point function for the different classes of probes. We close the paper by discussing these results and pointing out some further directions of research. We left some details to the appendices. Among them let us mention appendix-A since it includes a summary of our notations and YD terminology that is heavily used in this paper. We advise the reader to read it before reading the main part of the paper (sections 4 and 5).

## 2. Backgrounds and probes

Our central aim in this paper is to probe a class of “heavy” half-BPS states of type-IIB string theory on asymptotically  $\text{AdS}_5 \times \text{S}^5$  spacetimes, using “light” half-BPS states of the same theory. By heavy we mean states with energy/conformal weight that scales as  $N^2$ , whereas by light we mean states with energy/conformal weight that scales slower than  $N^2$ . Since on the gravity side, the heavy half-BPS states backreact on spacetime and generate the bulk geometry, an LLM geometry [25], we will call them the background states. The light half-BPS states on the other hand probe these geometries, hence we name them probes.

Our probe analysis will take place entirely in the dual conformal field theory, the  $\mathcal{N} = 4$   $\text{SU}(N)$  super Yang-Mills theory [26]. As is well known the half-BPS states of this theory can be described using YDs with at most  $N$  rows [23, 24], where the total number of its boxes is the conformal weight of the corresponding state. As a result, we will heavily use the YDs technology in this paper. For the needed notions, properties, as well as conventions used in this paper we refer the reader to appendix-A.

In this section, we will review the background states that we are interested in. We will discuss some of their main properties that will be crucial later on. After that, we will discuss the type of probes we will be using and some of their most important properties which will play a prominent role in sections 4 and 5.

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<sup>2</sup>Whether the actual group is  $\text{SU}(N)$  or  $\text{U}(N)$  is still a matter of debate. We expect that our leading large  $N$  result will not be modified if we use  $\text{SU}(N)$  group instead. See [21, 22] for details on the use of the  $\text{SU}(N)$  group and modifications to be brought to the  $\text{U}(N)$  formulas.

## 2.1 The superstar ensemble

In the following we review the superstar ensemble discussed in [1, section 3.3] in some detail. Our main interest is the main structure of the typical YDs of this ensemble. Hence, we will neither discuss the mapping between the LLM geometries [25] and the YDs of this ensemble, nor its effective description in terms of the superstar of [20]. We refer the interested reader to the paper [1]. We will start by describing the ensemble and its “average” YD. Then, we will discuss some general properties of typical states that will be of interest to us in the bulk of the paper. We will be following closely [1].

### 2.1.1 The ensemble and its limiting shape YD

The superstar ensemble is the set of YDs with fixed number of columns  $D$ , fixed number of rows  $N$  and fixed number of boxes  $\Delta$ , such that:

$$D \sim N, \quad \Delta = \frac{1}{2} N D \sim N^2, \quad (2.1)$$

which are weighed equally. Following [1], we will denote by  $r_i$  the length of row  $i$ , and by  $c_j$  the number of columns of length  $j$ <sup>3</sup>. In our conventions (see appendix-A), we have the relations:

$$c_N = r_N, \quad c_i = r_i - r_{i+1}; \quad 1 \leq i \leq N-1. \quad (2.2)$$

We can describe the superstar ensemble using a canonical ensemble. The associated partition function is given by:

$$\mathcal{Z} = \sum_{c_1, c_2, \dots, c_N=1}^{\infty} e^{-\beta \sum_{j=1}^N j c_j - \lambda \sum_{j=1}^N c_j} = \prod_{j=1}^N \frac{1}{1 - p q^j}, \quad (2.3)$$

where  $q = e^{-\beta}$ ,  $p = e^{-\lambda}$ ,  $\beta$  and  $\lambda$  are some positive parameters that will be fixed later on. Since we are dealing with the canonical ensemble instead of the microcanonical one, we need to fix the average of the number of boxes  $\Delta$  as well as the average of the number of columns  $D$  such that:

$$\Delta = \left\langle \sum_{j=1}^N j c_j \right\rangle = q \partial_q \log \mathcal{Z} = \sum_{j=1}^N \frac{j p q^j}{1 - p q^j} = \frac{1}{2} N D, \quad (2.4)$$

$$D = \left\langle \sum_{j=1}^N c_j \right\rangle = p \partial_p \log \mathcal{Z} = \sum_{j=1}^N \frac{p q^j}{1 - p q^j}. \quad (2.5)$$

We restrict ourselves to spelling out the results here leaving the details to appendix-C. We find that by fixing  $\beta$  and  $p$  as:

$$p = \frac{1 - q^D}{1 - q^{D+N}} \approx \frac{D}{D+N}, \quad \beta \sim \frac{1}{N}, \quad (2.6)$$

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<sup>3</sup>Notice that we have a different convention for numbering the YD rows than the one used in [1]. We start the numbering from top to bottom, whereas they number the way around, from bottom to top. This is the reason we have different expressions for  $r_i$  and  $c_i$  than them.

we satisfy the constraints above. The scaling of  $\log q$  with  $N$  is fixed by evaluating the entropy of this ensemble and using that  $\log q = (\partial S / \partial \Delta)$ .

When dealing with very large YDs that come with certain probability/weight, it is usually beneficial to construct the limit shape YD. This YD can be thought of as the “average” YD as it is constructed by finding a relation between the average length of a row and its position. In our present case, if we denote by  $y(x)$  the length of a row whose position is given by its number  $x$ , the limit shape YD turns out to be a triangle with the diagonal given by the equation:

$$y(x) = D \left( 1 - \frac{x}{N} \right) . \quad (2.7)$$

According to the usual intuition from statistical mechanics, most of the YDs in the superstar ensemble will be close to this limit shape YD. How close are they will be the subject of the next subsection.

### 2.1.2 Typical states

In the following we will summarize some of the main properties of typical YDs in the superstar ensemble that will be of importance to us. More precisely we want to know the deviation between a random typical YD  $\mathcal{O}$  and the limit shape YD  $\mathcal{O}_0$ . Since the number of boxes, columns and rows is held fixed in the superstar ensemble, the only thing that can happen is for boxes near the diagonal of  $\mathcal{O}_0$  to move from one row to another.

Let us first worry about boxes in the same row. A good estimate for the number of moved boxes is given by  $\sigma(D)$ . The reason being that for the limit shape YD there is a linear relation between  $D$  and the length of a row  $i$ ,  $\mathcal{O}_i = D(1 - i/N)$ . For a more precise treatment of  $\sigma(\mathcal{O}_i)$  see [1]. We have:

$$\sigma^2(D) = \sum_{i=1}^N (\langle c_i^2 \rangle - \langle c_i \rangle^2) = (p \partial_p)^2 \log \mathcal{Z} \approx \frac{D}{N} (D + N) \sim N . \quad (2.8)$$

Hence the fluctuation in the length of a row is of order  $\sqrt{N}$ . This is the usual thermal fluctuation since the length of almost all of the rows of the limit shape YD is of order  $N$ .

The other quantity that will be crucial to us is the total number of migrating boxes i.e. the total number of boxes that are moved around when comparing a typical YD in the superstar ensemble with its limit shape YD. A good estimate of that is the fluctuation in  $\Delta$ . We have:

$$\sigma^2(\Delta) = (q \partial_q)^2 \log \mathcal{Z} \approx \frac{1}{2} D (D + N) \sim N^2 . \quad (2.9)$$

Hence, the fluctuation in  $\Delta$  is of order  $N$ . Once again we find the usual thermal fluctuation.

Let us summarize what we found here. If we pick a random typical YD  $\mathcal{O}$  from the superstar ensemble, the length of a row in  $\mathcal{O}$  and the same row in the limit shape YD differ at worst by order  $\sqrt{N}$  boxes. If we add up the absolute values of these differences<sup>4</sup> we get a number that is at worst of order  $N$ .

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<sup>4</sup>Since the number of total boxes is held fixed in the superstar ensemble, these differences should add up to zero.

## 2.2 The probes and their $SU(N)$ and $S_h$ dimensions

After we discussed the most important properties of our background YDs, we turn our attention to the probe YDs. Their main property is that the total number of their boxes  $h$  is much less than  $N^2$  i.e.  $h \ll N^2$ . Another property of these YDs is that the total number of their rows  $n$  equals  $N$  at most. This is because we will be treating them as irreducible representations of  $U(N)$  when evaluating their two point function in our backgrounds, see section 3.1 for more details.

Among all possible probe YDs, we will limit our discussion in this paper to the *homogeneous* YDs. These are YDs where the ratio of the numbers of rows (columns) whose length does not scale with  $N$  in the same way as the number of columns  $d$  (respectively rows  $n$ ) to the total number of rows (respectively columns) tends to zero in the limit  $N \rightarrow \infty$ . If  $d$  denotes the total number of columns,  $n$  the total number of rows then we have:

$$h \sim n d . \quad (2.10)$$

For the other kinds of probes, we should think of them as the result of a tensor product of two or more homogeneous probes. In a sense, the homogeneous YDs are our building blocks that generate all the other YDs by the means of taking the tensor product between them. From now on, whenever we talk about a YD we mean a homogeneous one.

To fix the notation once and for all,  $\psi$  will stand for a probe YD,  $\psi_i$  the length of its  $i^{\text{th}}$  row,  $d$  the total number of its columns,  $n$  the total number of its rows,  $\psi_0 = \text{Max}\{d, n\}$ , and  $h$  the total number of its boxes. A collection of conventions to be used throughout this paper is collected in appendix-A.

In the remaining of this section we will study the leading behavior of  $\dim_N \psi$  the dimension of a probe YD  $\psi$  as an irreducible representation of  $SU(N)$  as well as  $\dim_h \psi$  its dimension as an irreducible representation of the permutation group  $S_h$ . These quantities will play an important role in this paper as they are intimately connected to the decomposition of the tensor product  $\mathcal{O} \otimes \psi$  (see section 4 for more details). The latter will play a role in the evaluation of the two point function (3.1) whose leading term we are after. Let us first start by giving two different formulas for the dimension of an  $SU(N)$  irreducible representation specified by a YD  $\psi$  (see for example [27, 28]). The first one is in terms of the difference in the lengths of different rows and reads:

$$\dim_N \psi = \prod_{k=1}^{N-1} \prod_{i=1}^{N-k} \left( 1 + \frac{\psi_i - \psi_{i+k}}{k} \right) . \quad (2.11)$$

The second formula is in terms of the hook lengths (see appendix-A). It reads:

$$\dim_N \psi = \frac{\prod_{i=1}^N \prod_{j=1}^{\psi_i} (N - i + j)}{\mathcal{H}_\psi} , \quad (2.12)$$

where  $\mathcal{H}_\psi = \prod_{i,j} h_{(i,j)}$ ,  $h_{(i,j)}$  is the hook length associated to the box  $(i, j)$ . Although we will heavily use the first expression, we will still need the second expression since it has a

similar form as the dimension of  $\psi$  as a representation of the permutation group  $S_h$ . The latter reads (see for example [27, 28]):

$$\dim_h \psi = \frac{h!}{\mathcal{H}_\psi}. \quad (2.13)$$

Since we are working with very large quantities, it is much useful to evaluate the log of these dimensions given that we are interested in the large  $N$  limit. Since the number of rows of  $\psi$  is given by  $n \leq N$ , we find using equation (2.11):

$$\log \dim_N \psi = \sum_{i=1}^{n-1} \sum_{j=1}^{n-i} \log \left( 1 + \frac{\psi_i - \psi_{i+j}}{j} \right) + \sum_{i=1}^n \sum_{j=n+1-i}^{N-i} \log \left( 1 + \frac{\psi_i}{j} \right). \quad (2.14)$$

It is clear that the leading term of the expression above will depend on how  $\psi_i$  scales with  $N$ . Such behavior leads us to classify the probes into three classes which are the following:

- **Generic probes class:** In this case both  $n$  and  $d$  are very small compared to  $N$  i.e.  $n, d \ll N$ . The reason we call them generic is that they exist for all regimes of  $h$  of interest to us ( $h \ll N^2$ ).
- **Linear probes class:** In this case either  $n$  or  $d$  but not both scales as  $N$ . The reason we coin them the name linear is that the leading behavior of the log of their  $SU(N)$  dimension is linear in  $h$  as we will see below. This class of probes is associated to the following regime of  $h$ :  $N \lesssim h \ll N^2$ .
- **Long probes class:** In this case  $d \gg N$ . Notice that this class of probes exists only in the regime  $h \gg N$ .

When evaluating the leading order of the log of the  $SU(N)$  and  $S_h$  dimensions of  $\psi$  in the following, we will discuss each class of probes on its own. For reasons that will be clear later on (see section 4), the leading behavior of  $\dim_h \psi$  will be of interest to us only if  $h \ll N$ . We will keep the discussion general and leave the treatment of a concrete example to appendix-D. But before continuing with the discussion of the dimensions of different probe classes, let us pause for a moment and discuss a curious duality of YDs<sup>5</sup> that will be useful below.

### 2.2.1 An approximate duality of Young diagrams

We know that two YDs that are related by the flip row  $\leftrightarrow$  column have the same  $S_h$  dimension. This is easily understood from equation (2.13) since the number of boxes as well as the hook length remain the same under such a flip. We will show in this section that the leading term of  $\log \dim_N \psi$  exhibits such an invariance for<sup>6</sup>  $d \leq N$ .

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<sup>5</sup>This looks like a manifestation of the hole/particle symmetry [29, 30] but not quite. This is because  $\dim_N \psi$  does not have a clear physical interpretation. However through its connection with the decomposition of the tensor product  $\mathcal{O} \otimes \psi$  (see section 4) it has a connection with the aforementioned symmetry.

<sup>6</sup>This is because  $n \leq N$  as a result of the YD being an  $U(N)$  representation.



We will assume in the following that  $d \ll n$ . This assumption is not restrictive since we know that such YDs are one end of this duality. The only non-covered case is when  $d \sim n$ , but as we will see later on, this case fits nicely in the manipulations used for  $d \ll n$ . A property that will be used below, which will be argued for in subsections 2.2.2 and 2.2.3 below and further checked in a concrete example in appendix-D, is that the leading behavior of  $\log \dim_N \psi$  is either of order  $h \log N$  or at worst of order  $h$  for the cases of interest to us. We will take these properties as granted for now.

Our starting point is the relation (2.12) in the case  $d \ll n$ . First, we rewrite it as:

$$\log \dim_N \psi = \sum_{i=1}^{N-1} \sum_{j=1}^{\psi_i} \log(N - \psi_i + i + j) - \sum_{i=1}^{N-1} \sum_{j=1}^{\psi_i} \log h_{(i,j)} . \quad (2.15)$$

The duality in this formula is a statement about the exchange  $i \leftrightarrow j$ . The part that depends only on the hook length is trivially invariant under such a change. The quantity  $(i + j)$  is also invariant. The only problematic part is  $\psi_i$ . However its contribution is much smaller than  $h$  and hence subleading given our claim above. To prove this claim, we use that  $\psi_i \ll N$  to expand the log term and get:

$$\delta \log \dim_N \psi \approx \sum_{i=1}^{N-1} \sum_{j=1}^{\psi_i} \frac{\psi_i}{N + i + j} < \sum_{i=1}^{N-1} \frac{\psi_i^2}{N + i} < d^2 \ll h ,$$

where we used that  $h \sim nd$ . Notice that these manipulations do not work if  $d \gg n$ . But this is not a problem since this regime will be at the other end of the duality for the case  $n \gg d$ .

As already advanced in the beginning, the same manipulations lead to the same conclusion in the case  $n \sim d$ . The reason being that in this case both  $n$  and  $d \ll N$  since we are interested in cases where  $h \ll N^2$ . Hence, these kind of YDs belong to the generic class. Although the correction that we get from the terms depending on  $\psi_i$  will be of order  $h$ , the leading behavior of  $\log \dim_N \psi$  is of order  $h \log N$  (see the subsection below, and also appendix-D). Hence the duality survives in this case as well.

### 2.2.2 The generic probes class

In this case we have  $n \ll N$  and  $\psi_i \ll N$ . It turns out that we need to distinguish between two cases:  $n \ll d$ , and  $d \ll n$ . The case  $d \sim n$  is a trivial consequence of the previous two cases. The easiest case to deal with is when we have  $n \ll d$ . Let us discuss it first, then turn to the second case  $d \ll n$ . In the case  $n \ll d$  (or  $n^2 \ll h$ ), we have from equation (2.14) the following upper bound:

$$\begin{aligned} \log \dim_N \psi &\leq \sum_{i=1}^n \sum_{j=1}^{N-i} [\log(j + \psi_i) - \log j] \\ &\lesssim \sum_{i=1}^n [(N - i + \psi_i) \log(N - i + \psi_i) - (N - i) \log(N - i) - \psi_i \log \psi_i] \\ &\lesssim \sum_{i=1}^n [\psi_i \log N + \psi_i - \psi_i \log \psi_i] \approx h \log \left( \frac{N}{d} \right) , \end{aligned} \quad (2.16)$$

where in the first line we used that  $(\psi_i - \psi_{i+j}) < \psi_i$  for  $1 \leq i < n$ . To move from the first to the second line we used equation (B.6) to evaluate the sum over  $j$ , and to get the third line we used that  $\psi_i \ll N$  and  $n \ll N$ . The last result is a consequence of:

$$\sum_{i=1}^n \psi_i \log \psi_i - h \log d = \sum_i^n \psi_i \log \left( \frac{\psi_i}{d} \right) \sim \sum_{i=1}^n \psi_i \sim h, \quad (2.17)$$

where we used the fact that the ratio  $(\psi_i/d)$  is independent of  $N$  for almost all of the rows.

For the lower bound of (2.14), we get:

$$\begin{aligned} \log \dim_N \psi &\geq \sum_{i=1}^n \sum_{j=n+1-i}^{N-i} [\log(j + \psi_i) - \log j] \\ &\gtrsim h \log N + h - \sum_{i=1}^n [(n - i + \psi_i) \log(n - i + 1 + \psi_i) - (n - i) \log(n - i + 1)], \end{aligned}$$

where in the first line we use that the first sum in (2.14) is a positive number. To get the last line, we used the same steps as above. We need to deal with the last sum. We find using that  $\psi_i \sim d \gg n$  the following approximate value:

$$\sum_{i=1}^n [(n - i + \psi_i) \log(n - i + 1 + \psi_i) - (n - 1) \log(n - i + 1)] \approx \sum_{i=1}^n \psi_i \log \psi_i \approx h \log d,$$

which we discussed before. Plugging this result in the expression for  $\log \dim_N \psi$  above, we find:

$$\log \dim_N \psi \gtrsim h \log \left( \frac{N}{d} \right).$$

Combining this lower limit with the upper limit in (2.16), we conclude that the leading behavior of  $\log \dim_N \psi$  in the case  $N \gg d \gg n$  is given by:

$$\log \dim_N \psi \approx h \log \left( \frac{N}{d} \right). \quad (2.18)$$

What about the other case  $n \gg d$ ? For this we take advantage of the duality discussed in section-2.2.1 to swap the rows and columns of  $\psi$ , which brings us to the previous case where  $n$  here plays the role of  $d$  there and vice versa. Hence, in the case  $n \gg d$ , the leading behavior of  $\log \dim_N \psi$  is given by:

$$\log \dim_N \psi \approx h \log \left( \frac{N}{n} \right).$$

As a conclusion, we find that the leading behavior of  $\log \dim_N \psi$  in the large  $N$  limit in the case where  $n$  and  $d \ll N$  is given by:

$$\log \dim_N \psi \approx h \log \left( \frac{N}{\psi_0} \right), \quad (2.19)$$

where  $\psi_0 = \text{Max}\{d, n\}$ .

The next quantity we want to find is the leading term of  $\log \dim_h \psi$  in the case  $h \ll N$ . The only non-trivial term in the dimension relation (2.13) is  $\mathcal{H}_\psi$ . Its leading behavior can be derived using the expression (2.12) together with the leading behavior (2.19). First, we need to deal with the numerator of (2.12), we have:

$$\begin{aligned}
\log \text{num}_N \psi &= \sum_{i=1}^n \sum_{j=1}^{\psi_i} \log(N + j - i) \\
&\approx \sum_{i=1}^n [(N - i + \psi_i) \log(N - i + \psi_i) - (N - i) \log(N - i) - \psi_i] \\
&\approx \sum_{i=1}^n \psi_i \log(N - i) \approx h \log N,
\end{aligned} \tag{2.20}$$

where we use the approximation (B.6) to get the second line, then we used that  $\psi_i \ll N$  and  $n \ll N$  to approximate the sums in the second and third line respectively. Next, we plug the leading behavior of  $\log \dim_N \psi$  given in equation (2.19) and the leading behavior of the numerator of (2.12) that is given in (2.20) above, in the equation (2.12), to get the following leading behavior of the product over the hook lengths:

$$\log \mathcal{H}_\psi \approx h \log \psi_0,$$

where  $\psi_0 = \text{Max}\{d, n\}$ . As a result, the expression (2.13) for the dimension of  $\psi$  as a representation of  $S_h$  leads to the following leading behavior:

$$\log \dim_h \psi \approx h \log \left( \frac{h}{\psi_0} \right). \tag{2.21}$$

Notice that in the case where  $h \sim \psi_0$ , the leading behavior of  $\log \dim_h \psi$  will be proportional to  $h$ . We will mostly use the expression above for the leading behavior of the  $S_h$  dimension of this class of probes, keeping in mind the special case  $h \sim \psi_0$ .

This is the only class of probes where we will need the leading behavior of their dimension as a representation of  $S_h$ . In the remaining of this section we will look for the leading behavior of the  $\text{SU}(N)$  dimension of YD in the other two classes. Although a universal exact expression for the leading term is not always possible, we will derive its leading behavior in the worst situations.

### 2.2.3 The linear probes class

In this class of probes, we have either  $d \sim N$  or  $n \sim N$ . Following the same route as above, we first discuss the case  $d \sim N$ , then use the duality discussed in section-2.2.1 for the case  $n \sim N$ . The reason we can use this duality will be clear below.

Our starting point is once again the relation (2.14). We have in the case  $d \sim N \gg n$

the following upper bound:

$$\begin{aligned}
\log \dim_N \psi &\leq \sum_{i=1}^n \sum_{j=1}^{N-i} [\log(j + \psi_i) - \log j] \\
&\lesssim \sum_{i=1}^n [(N - i + \psi_i) \log(N + \psi_i) - (N - i) \log N - \psi_i \log \psi_i] \\
&\lesssim \sum_{i=1}^n [(N + \psi_i) \log(1 + \bar{\psi}_i) - \psi_i \log \bar{\psi}_i] \sim h,
\end{aligned}$$

where we used the approximation (B.6) to evaluate the sum over  $j$  in the first line and that  $n \ll N$  to get the second line. In the last line we introduced the quantity  $\bar{\psi}_i = \psi_i/N$ , which is independent of  $N$ . At the end we used that  $\psi_i \sim d \sim N$  and  $h \sim n d \sim nN$ . The reason we did not try to get an exact result here is that knowing that the upper bound is of order  $h$  is more than enough for our purposes, see section 5 for more details.

To complete the circle of thoughts, although not needed, let us look for a lower bound on the leading behavior of  $\log \dim_N \psi$ . We have:

$$\begin{aligned}
\log \dim_N \psi &\geq \sum_{i=1}^n \sum_{j=n+1-i}^{N-i} [\log(j + \psi_i) - \log j] \\
&\gtrsim \sum_{i=1}^n [(N - i + \psi_i) \log(N + \psi_i) - (N - i) \log N] \\
&\quad - \sum_{i=1}^n [(n - i + \psi_i) \log \psi_i - (n - i) \log(n - i)] \\
&\gtrsim \sum_{i=1}^n [(N + \psi_i) \log(1 + \bar{\psi}_i) - \psi_i \log \bar{\psi}_i] \sim h,
\end{aligned}$$

Where we used the same steps as in the derivation above. Notice that we get the same expression as for the upper bound. Hence we conclude that in the case  $d \sim N$ , we have the following leading behavior of  $\log \dim_N \psi$ :

$$\log \dim_N \psi \approx \sum_{i=1}^n [(N + \psi_i) \log(1 + \bar{\psi}_i) - \psi_i \log \bar{\psi}_i] \sim h, \quad (2.22)$$

which is of order  $h$  as argued above.

This was for the case  $d \sim N$ , what about the other possibility  $n \sim N$ . Once again we take advantage of the duality discussed in section-2.2.1 to safely conclude that the leading behavior of the dimension in this case ( $n \sim N$ ), is given by:

$$\log \dim_N \psi \approx \sum_{i=1}^n [(N + n_i) \log(1 + \bar{n}_i) - n_i \log \bar{n}_i] \sim h, \quad (2.23)$$

where  $n_i$  stands for the length of the columns of  $\psi$  and  $\bar{n}_i = n_i/N$ .

All in all, we conclude that the leading behavior of the dimension of the linear probes is such that:

$$\log \dim_N \psi \sim h , \quad (2.24)$$

where the actual value of the non-zero coefficient that multiplies  $h$  is not important to us.

### 2.2.4 The long probes class

The shape of these probes suggest that their dimension will be subleading with respect to the previous two cases. This is because  $\mathcal{H}_\psi$  in (2.12) will be of the same order as the numerator. Let us proceed and check this intuitive guess. As usual we are going to look for an upper and a lower bounds of  $\log \dim_N \psi$  using equation (2.14). For the upper bound, we find:

$$\begin{aligned} \log \dim_N \psi &\leq \sum_{i=1}^n \sum_{j=1}^{N-i} [\log(j + \psi_i) - \log j] \\ &\lesssim \sum_{i=1}^n [(N - i + \psi_i) \log(N - i + \psi_i) - \psi_i \log \psi_i - (N - i) \log(N - i)] \\ &\lesssim \sum_{i=1}^n [N \log \psi_i - N \log N] \approx n N \log \left( \frac{d}{N} \right) , \end{aligned}$$

where we used as usual (B.6) to evaluate the sum over  $j$  to get the second line. To get the third line we used that  $\psi_i \gg N \gg n$ . To get the last result we used that:

$$n \log d - \sum_{i=1}^n \log \psi_i = \sum_{i=1}^n \log \frac{d}{\psi_i} \sim n ,$$

since the ratio  $(\psi_i/N)$  is  $N$ -independent for most  $\psi_i$ .

What about the lower bound? Using the same manipulations as above, we easily find:

$$\begin{aligned} \log \dim_N \psi &\geq \sum_{i=1}^n \sum_{j=n+1-i}^{N-i} [\log(\psi_i + j) - \log j] \\ &\gtrsim \sum_{i=1}^n [(N - i + \psi_i) \log(N - i + \psi_i) - (n - i + \psi_i) \log(n - i + \psi_i)] \\ &\quad - \sum_{i=1}^n [(N - i) \log(N - i) - (n - i) \log(n - i)] \\ &\gtrsim \sum_{i=1}^n [N \log \psi_i - N \log N] \approx n N \log \left( \frac{d}{N} \right) . \end{aligned}$$

Combining this result with the upper bound above, we conclude that the leading behavior of the  $SU(N)$  dimension of these probes is given by:

$$\log \dim_N \psi \approx n N \log \left( \frac{d}{N} \right) . \quad (2.25)$$

Notice that by exchanging the roles of  $N$  and  $d$ , one can map this leading term to the corresponding one in the case of generic probes given in (2.19), taking into account that  $h \sim n d$ .

### 3. Probes in $\text{AdS}_5 \times \text{S}^5$

The states we are dealing with in this paper are half-BPS states of type-IIB string theory on asymptotic  $\text{AdS}_5 \times \text{S}^5$ . These states can be described on the dual field theory side using YDs seen as irreducible representation of  $\text{U}(N)$ , where  $N$  is the number of fluxes in the background geometry [23, 20]. We are mainly interested in probing the backgrounds associated to the microstates of the superstar of [20] according to the proposal advanced in [1]. Remember that these microstates are characterized by YDs whose number of columns  $N_c$ , number of rows  $N_r$  and number of boxes  $\Delta$  are all held fixed as follows:

$$N_r = N, \quad N_c = D \sim N, \quad \Delta = \frac{1}{2} N N_c,$$

see subsection 2.1 and [1] for more details. In this section we will spell out the probing tool that we will be using in this paper in its full generality. After that, we will warm up with a toy model that describes a background that does not belong to the superstar ensemble for reasons that will be clear later on.

#### 3.1 The two point function

The game we will be playing in this paper is as follows. We have a background that is generated by a typical state  $\mathcal{O}$  of the superstar ensemble and we add on top of that a light probe  $\psi$ , then see what happens. According to [23], modulo an overall trivial space time dependence which is completely fixed by conformal symmetry, everything boils down to evaluating the two point function:

$$\langle\langle\psi\psi\rangle\rangle_{\mathcal{O}} = \frac{\langle(\mathcal{O} \otimes \psi)(\mathcal{O} \otimes \psi)\rangle}{\langle\mathcal{O}\mathcal{O}\rangle}, \quad (3.1)$$

where  $\psi$  is our probe,  $\mathcal{O}$  is our background, and the vacuum two point function  $\langle\alpha\beta\rangle$  is given by:

$$\langle\alpha\beta\rangle = \delta_{\alpha\beta} \prod_{i,j} (N - i + j), \quad (3.2)$$

where  $\delta_{\alpha\beta}$  is a schematic notation that means that the two YDs  $\alpha$  and  $\beta$  should be identical, and the product is over all the boxes of the YD  $\alpha$  where  $i$  is the column number and  $j$  is the row number. The expression (3.1) is evaluated as follows. We first decompose the tensor product  $\mathcal{O} \otimes \psi$  into irreducible representations of  $\text{U}(N)$ :

$$\mathcal{O} \otimes \psi = \bigoplus_k d_k \varphi_k, \quad (3.3)$$

where  $d_k$  is the degeneracy of the YD  $\varphi_k$ , and  $k$  is a summation index that will not play any important role in the following. Next, we use that the two-point  $\langle\mathcal{O}\mathcal{O}'\rangle$  is bilinear to find:

$$\langle\langle\psi\psi\rangle\rangle_{\mathcal{O}} = \sum_k d_k^2 \langle\varphi_k\varphi_k\rangle_{\mathcal{O}}, \quad (3.4)$$

where we used that the two point function (3.2) is diagonal and introduced the notation:

$$\langle \bullet \bullet \rangle_{\mathcal{O}} = \frac{\langle \bullet \bullet \rangle}{\langle \mathcal{O} \mathcal{O} \rangle} , \quad (3.5)$$

so that we do not get a cluttered expression. We will be using this simplified notation from now on. So our task can be summarized into the following steps:

1. Get the needed information from the tensor product decomposition (3.3). These include the degeneracy  $d_k$ , the type of YDs  $\varphi_k$ , and their total number. From now on, we will refer to the tensor product of two irreducible representations of  $SU(N)$  by the tensor product between the associated YDs, in an abuse of language.
2. Evaluate the term  $\langle \varphi_k \varphi_k \rangle_{\mathcal{O}}$  in the expression (3.4).
3. Finally collect all the intermediate results to get the final answer.

We will deal with each step on its own in the following sections. Some of the details will be left to appendices that we will refer to at the right places. But before doing so, let us discuss a simple toy model. The results obtained here will serve as a good reference point later on.

### 3.2 A toy model: A background outside the superstar ensemble

The decomposition of the tensor product  $\mathcal{O} \otimes \psi$  is very involved in general (see subsection 4.1.1), but there are some simple situations where things become straightforward. One of the simplest cases correspond to a background YD  $\mathcal{Q}$  where all of its  $N$  rows have the same length:

$$\mathcal{Q}_i = \kappa N ,$$

where  $\kappa$  is an arbitrary constant. Although this background  $\mathcal{Q}$  is not of immediate importance to us since it is not part of the superstar ensemble, it is both a good warm up exercise as well as a good reference point for the two point function (3.1) of the superstar ensemble.

The simplicity of the background  $\mathcal{Q}$  resides in the fact that there is only one YD  $\varphi_0$  when decomposing the tensor product  $\mathcal{Q} \otimes \psi$ . If we denote by  $\delta_i$  the number of boxes added to the  $i^{\text{th}}$  row of  $\mathcal{Q}$  to form the YD  $\varphi_0$  then we have:

$$\delta_i = \psi_i .$$

The two point function (3.1) simplifies drastically in this case as we need to only evaluate  $\langle \varphi_0 \varphi_0 \rangle_{\mathcal{Q}}$ . The latter can be easily evaluated using its defining equation (3.5) together with the explicit expression of the vacuum two point function  $\langle \alpha \beta \rangle$  given by (3.2). Using theses equations we find that:

$$\log \langle \varphi_0 \varphi_0 \rangle_{\mathcal{Q}} = \sum_{i=1}^N \sum_{j=1}^{\delta_i} \log(N + \mathcal{Q}_i + j - i) , \quad (3.6)$$

where  $\mathcal{Q}_i$  stands for the length of row  $i$  of the YD  $\mathcal{Q}$ . We can easily evaluate the sum over  $j$  using equation (B.6) to get the following approximate expression:

$$\begin{aligned} \log \langle \varphi_k \varphi_k \rangle_{\mathcal{Q}} &\approx -h + \sum_{i=1}^N (N + \mathcal{Q}_i + \delta_i - i) \log(N + \mathcal{Q}_i + \delta_i - i) \\ &\quad - \sum_{i=1}^N (N + \mathcal{Q}_i - i) \log(N + \mathcal{Q}_i - i) , \end{aligned} \quad (3.7)$$

where we used  $\sum_{i=1}^k \delta_i = h$ . Let us now specify the formula to our case where  $\mathcal{Q}_i = \kappa N$  and  $\delta_i = \psi_i$  nonzero only for  $i \leq n$ . We get:

$$\begin{aligned} \log \langle \varphi_0 \varphi_0 \rangle_{\mathcal{Q}} &\approx -h + \sum_{i=1}^n [(1 + \kappa)N + \psi_i - i] \log[(1 + \kappa)N + \psi_i - i] \\ &\quad - \sum_{i=1}^n [(1 + \kappa)N - i] \log[(1 + \kappa)N - i] . \end{aligned}$$

To proceed further, we need to distinguish between different probe classes.

**The generic probes class** In this class we have both  $\psi_i \ll N$  and  $n \ll N$ . We find after expand the log:

$$\log \langle \langle \psi \psi \rangle \rangle_{\mathcal{Q}} = \log \langle \varphi_0 \varphi_0 \rangle_{\mathcal{Q}} \approx h \log N . \quad (3.8)$$

**The linear probes class** In this class we have either  $d \sim N$  or  $n \sim N$ . In the case  $n \sim N$ , we have  $d \ll N$  and hence we can use the same manipulations as before to get:

$$\log \langle \langle \psi \psi \rangle \rangle_{\mathcal{Q}} \approx h \log N . \quad (3.9)$$

In the opposite case  $\psi_i \sim N$ , we have  $n \ll N$  so we can expand the log term around  $i = 0$ . We get in this case the same result as above.

**The long probes class** In this class we have  $d \gg N$  and  $n \ll N$ . Expanding the log term appropriately, we get:

$$\log \langle \langle \psi \psi \rangle \rangle_{\mathcal{Q}} \approx h \log d . \quad (3.10)$$

Notice this result does not depend explicitly on  $N$ . Let us now turn to the more interesting case where the background  $\mathcal{O}$  is a typical state in the superstar ensemble.

#### 4. Dealing with the tensor product $\mathcal{O} \otimes \psi$

Our starting point in evaluating the two point function (3.1) is to construct the tensor product decomposition (3.3). For our purposes, we do not need to get all the details of this decomposition which is hopeless. We only need to get an estimate of the degeneracies  $d_k$  of the YDs  $\varphi_k$ , an estimate of the total number of these YDs  $d_t = \sum_k d_k$ , and a rough idea on the shape of the YDs  $\varphi_k$  so that we can evaluate the quantities  $\langle \varphi_k \varphi_k \rangle_{\mathcal{O}}$ . In the following we will start by reviewing the construction of the YDs that appear in the decomposition



of the tensor product of two YDs of  $U(N)$ . By studying carefully the conditions on these YDs that appear in such decomposition, we propose a map between semi-standard Young tableaux (SSYT<sub>x</sub>) and these YDs. Next, using this map we discuss the shape of “dominant” YDs  $\varphi_k$  and their degeneracies  $d_k$  according to the scaling of  $h$  with  $N$ . Some of the details will be left to appendices E and F.

#### 4.1 The tensor product decomposition and the semi-standard Young tableaux

The first question to answer is how to construct the YD  $\varphi_k$  starting from the YD  $\mathcal{O}$  and  $\psi$  when the tensor product  $\mathcal{O} \otimes \psi$  is seen as a tensor product between two irreducible representation of  $U(N)$ ? Fortunately there is a well known recipe to construct the decomposition of the tensor product of two YDs. We will first review this recipe, then we will repack the information about such decomposition into labelings of  $\psi$  in the case of our interest.

##### 4.1.1 Decomposing the tensor product of two representations

Let  $A$  and  $B$  be two YDs associated to two irreducible representations of  $U(N)$ . The YDs corresponding to the irreducible representations appearing in the decomposition of the tensor product  $A \otimes B$  are constructed by adding all the boxes of one of the two YDs, say  $B$ , to the other YD, here  $A$ , in all possible ways subject to the following rules (see for example [31, Chapter.9]):

- First, we fill each box of the YD  $B$  with a label  $a_i$ , where  $i$  stands for the number of the row the box belongs to.
- We start by adding the left most box<sup>7</sup> from the first row of  $B$ , which carries the label  $a_1$ , to the YD  $A$  in all possible position such that we end up a YD of  $U(N)$  i.e. the length of rows is decreasing from top to bottom, and the length of each column is at most  $N$ . We repeat the same process with the remaining boxes in the first row of  $B$  starting from left to right, keeping in mind that each new added box should be to the right or below the previous one and that no two boxes among the added ones are in the same column. The latter two conditions are solely for boxes that belong to the same row in  $B$ .
- We repeat exactly the same process for each row of the YD  $B$  until we finish all of its boxes.
- Finally, we keep only the YDs that satisfy the following rule. Let  $C$  denote one of the resultant YDs. We start our journey at the upper right box of the YD  $C$  going from the right to the left of the first row, then move down to the second row and, once again, start from its right most box and move to the left, and so on and so forth until we reach the lower left box of  $C$ . At each box in this journey, the number of the

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<sup>7</sup>This is a matter of choice as boxes belonging to the same row are treated on equal footing, hence the reason they carry the same label. In most literatures one starts from the right most box, but we choose to start from the left most one for reasons to be clear later on.

newly added boxes with label  $a_i$  should not exceed the number of boxes with label  $a_j$  if  $j < i$ .

At the end we collect the resulting YDs according to their shape. Since the decomposition of the tensor product will play a crucial role in this paper, it is a good idea to name the rules in the construction above, instead of trying to explain them each time. Essentially, we have three important rules which are as follows. The *YD* rule will refer to the condition that the resultant digram should be a YD at each step of adding a new box. The *antisymmetry* rule will refer to the condition that two boxes originating from the same row of the YD  $B$  should not belong to the same column in the resulting YDs. The *ordering* rule will refer to the combination of the order in which we add boxes originating from the same row in the YD  $B$  (explained in the first step above) which will be called the *row* rule from now on, and the last condition above which will be called the *column* rule from now on.

Before continuing, let us apply these rules in the case of the following tensor product:

$$\begin{aligned}
\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline a & a \\ \hline b & \\ \hline \end{array} &= \left( \begin{array}{|c|c|c|} \hline \cdot & \cdot & a \\ \hline \cdot & & \\ \hline \cdot & & \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline \cdot & \cdot \\ \hline \cdot & a \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline \cdot & \cdot \\ \hline \cdot & \\ \hline a & \\ \hline \end{array} \right) \otimes \begin{array}{|c|} \hline a \\ \hline b \\ \hline \end{array} \\
&= \left( \begin{array}{|c|c|c|c|} \hline \cdot & \cdot & a & a \\ \hline \cdot & & & \\ \hline \cdot & & & \\ \hline \end{array} \oplus \begin{array}{|c|c|c|} \hline \cdot & \cdot & a \\ \hline \cdot & & \\ \hline \cdot & & \\ \hline \end{array} \oplus \begin{array}{|c|c|c|} \hline \cdot & \cdot & a \\ \hline \cdot & & \\ \hline a & & \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline \cdot & \cdot \\ \hline \cdot & a \\ \hline a & \\ \hline \end{array} \right) \otimes \begin{array}{|c|} \hline b \\ \hline \end{array} \\
&= \begin{array}{|c|c|c|c|} \hline \cdot & \cdot & a & a \\ \hline \cdot & b & & \\ \hline \end{array} \oplus \begin{array}{|c|c|c|c|} \hline \cdot & \cdot & a & a \\ \hline \cdot & & & \\ \hline b & & & \\ \hline \end{array} \oplus \begin{array}{|c|c|c|c|} \hline \cdot & \cdot & a & \\ \hline \cdot & & & \\ \hline \cdot & a & b & \\ \hline \end{array} \oplus \begin{array}{|c|c|c|} \hline \cdot & \cdot & a \\ \hline \cdot & & \\ \hline b & & \\ \hline \end{array} \oplus \begin{array}{|c|c|c|} \hline \cdot & \cdot & a \\ \hline \cdot & b & \\ \hline a & & \\ \hline \end{array} \\
&\quad \oplus \begin{array}{|c|c|c|} \hline \cdot & \cdot & a \\ \hline \cdot & & \\ \hline a & & \\ \hline b & & \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline \cdot & \cdot \\ \hline \cdot & a \\ \hline a & b \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline \cdot & \cdot \\ \hline \cdot & a \\ \hline a & b \\ \hline \end{array} \\
&= \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} \oplus \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} \oplus \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} \oplus 2 \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} \\
&\quad \oplus \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline & \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline & \\ \hline \end{array} . \tag{4.1}
\end{aligned}$$

In this example we used letters to label the boxes, which is fine since we are still following the rules. Notice that in this example we have a single YD that appears twice with different order of the letters  $a$ ,  $b$ , hence it is doubly degenerate. Notice also that the box with the letter  $b$  never appears in the first row. This is a straightforward consequence of the ordering rule. In general, boxes from row  $i$  in  $B$  cannot be attached to the row  $j$  in  $A$  if  $j < i$ .

There is another immediate consequence of the ordering rule which has to do with the number of added boxes to a row  $i$  in  $A$ . Let us denote by  $n_i^j$  the number of boxes from row  $j$  in  $B$  that are added to row  $i$  in  $A$ . Suppose that  $i$  is the row we are interested in, and suppose also that the added boxes to this row belong to the rows  $m \leq k \leq n$  from  $B$ . We

have the following conditions:

$$n_i^k \leq \sum_{j=1}^{i-1} (n_j^{k-1} - n_j^k) ; \quad \forall k , \quad m+1 \leq k \leq n .$$

Summing these relations gives us:

$$\sum_{k=m}^n n_i^k \leq n_i^m + \sum_{j=1}^{i-1} (n_j^m - n_j^n) < b_m , \quad (4.2)$$

where  $b_m$  is the length of row  $m$  in  $B$ . Hence, if the boxes added to row  $i$  of the YD  $A$  used to be in rows of the YD  $B$  with  $m$  denoting the number of the upper row among them, then the number of added boxes to the row  $i$  is smaller or equal the number of boxes in the row  $m$  of the YD  $B$ . These two properties will have a nice interpretation after introducing our map between the YD  $\varphi_k$  and SSYT  $\psi$ . See appendix-E for more details.

Our next target after we learned how to decompose a tensor product into irreducible representations (YDs) is to count the number of the resulting YDs and their degeneracies. This will be the subject of the next few sections. We will first start by encoding the information about YDs that appear in the tensor product decomposition of  $\mathcal{O} \otimes \psi$  into labelings of  $\psi$ . Then, using this new way of characterizing the tensor product decomposition, we proceed to deal with our main target, estimating the number of YD  $\varphi_k$  and their degeneracies  $d_k$ .

#### 4.1.2 Repackaging the information about the tensor product decomposition

Our aim in this paper is to study the two point function (3.1) as we vary the probe  $\psi$ . Hence, it will be very effective and fruitful to use the YD  $\psi$  in packaging the information about the tensor product  $\mathcal{O} \otimes \psi$ . It is clear from the way we construct the decomposition of the tensor product of two YDs that it is much easier in the present case  $\mathcal{O} \otimes \psi$  to distribute the boxes of  $\psi$  on  $\mathcal{O}$ . A naive guess for packaging the information about the YDs  $\varphi_k$  that will appear in the decomposition of  $\mathcal{O} \otimes \psi$ , is to associate to each box of  $\psi$  the number of the row of the YD  $\mathcal{O}$  it is attached to. As a result, we get for each  $\varphi_k$  a certain labeling of  $\psi$ . To get more information about the type of these labelings, we need to answer the following question: What are the implications of the tensor product decomposition rules that we discussed previously on the labelings of  $\psi$ ?

The easiest condition to apply on the possible labelings of  $\psi$  is that the number of rows of  $\varphi_k$  should not exceed  $N$ . Since the YD  $\mathcal{O}$  has already  $N$  rows, see section 2.1, the labels of  $\psi$  should be among the numbers  $\mathcal{L} = \{1, 2, \dots, N\}$ . For a specific  $\varphi_k$  the collection of the labels of the associated labeling of  $\psi$  will in general be a subset of  $\mathcal{L}$ . This observation will play an important role later on. What about the impact of the three rules: the YD rule, the antisymmetry rule, and the ordering rule? It is clear that the last rule, the ordering rule, is universal in the sense that its outcome does not depend on the YD  $\mathcal{O}$  in contrast to the other two rules whose outcomes depend highly on which background  $\mathcal{O}$  we are studying. Hence, it is a good idea to look first for the implication of the ordering rule on the possible labelings of  $\psi$ , then look for possible corrections due to the other two

rules. In the remaining of this subsection, we will only discuss the impact of the ordering rule on the labelings of  $\psi$  leaving the inclusion of the other two rules mainly to appendix-F.

The ordering rule as defined previously (section 4.1.1) has two parts to it. The first part, called the row rule, has to do with boxes that belong to the same row of  $\psi$  whereas the second one, called the column rule, has to do with boxes that belong to different rows of  $\psi$ . It is easy to see that the row rule is equivalent to the requirement that in the associated labeling of  $\psi$ , the numbers in the same row should be weakly increasing from left to right. What about the column rule? To understand its consequence, let us assume for a moment that  $\psi$  is a YD with a single column. To simplify the discussion below we will also number the boxes of  $\psi$  according to their row i.e. box  $i$  means a box in row  $i$  of  $\psi$ . It is easy to see that in this case, the column rule is equivalent to the condition that the box  $i$  should not be added to the rows of  $\mathcal{O}$  that are before the one that has box  $j$  added to it if  $j < i$ . Hence, the acceptable labelings of  $\psi$  in this case should be such that the numbers labeling boxes of the single column are strictly increasing from top to bottom. What about the case where  $\psi$  has two columns? Let us concentrate on the four boxes belonging to rows  $i$  and  $j$  where  $j > i$ . The column rule implies that the row to which the box  $j$  from the first column of  $\psi$  is attached to, should be below the row to which the box  $i$  from the first column of  $\psi$  is attached to. The same condition should be satisfied by the boxes  $i$  and  $j$  from the second column of  $\psi$ . Hence, in this case the column rule requires that the number labeling boxes in the same column of  $\psi$  to be strictly increasing from top to bottom for an acceptable labeling of  $\psi$ . It is not hard to see that this is the manifestation of the column rule in the acceptable labelings of a generic  $\psi$ . This can be easily seen by iterating the previous discussion for each row of  $\psi$ .

In our previous example of the tensor product decomposition (4.1), the labelings of the resultant YDs are:

$$\begin{array}{|c|c|}, \begin{array}{|c|c|}, \begin{array}{|c|c|}, \left( \begin{array}{|c|c|}, \begin{array}{|c|c|} \right), \begin{array}{|c|c|}, \begin{array}{|c|c|}, \begin{array}{|c|c|} \end{array} \end{array} \quad (4.3)$$

which clearly satisfies the last two conditions: numbers in the same row are weakly increasing from left to right, and numbers in the same column are strictly increasing from top to bottom. The two YT<sub>x</sub> between bracket correspond to the degenerate YD.

To summarize, the shape of  $\mathcal{O}$ , the fact that we are dealing with the  $U(N)$  group together with the ordering rule implies that the acceptable labelings of  $\psi$  that are associated to a YD  $\varphi_k$  in the decomposition of  $\mathcal{O} \otimes \psi$  should be such that:

- The labels belong the set of numbers  $\mathcal{L} = \{1, 2, \dots, N\}$ .
- The numbers labeling boxes in the same row should be weakly increasing from left to right.
- The numbers labeling boxes in the same column should be strictly increasing from top to bottom.

These last two conditions define the so called SSYT, see appendix-A. If we take into account the first condition, the number of these SSYT $\psi$  is the dimension of  $\psi$  as a representation of  $SU(N)$ , see for example [27]. Since we are not taking into account neither the YD rule nor the antisymmetry rule,  $\dim_N \psi$  is an upper bound on the total number of the YDs  $\varphi_k$  i.e.  $d_t = \sum_k d_k \leq \dim_N \psi$ .

Parameterizing the decomposition of the tensor product  $\mathcal{O} \otimes \psi$  in terms of SSYT $\psi$  gave us an idea about the total number of the YDs  $\varphi_k$ , but what we are really after is the value of the degeneracies  $d_k$  and the range of the index  $k$ . Let us pick a YD  $\varphi_k$ . This YD is fixed once we know the  $N$ -tuple  $\beta = (n_1, n_2, \dots, n_N)$ , where  $n_i$  stands for the added boxes to row  $i$  of the YD  $\mathcal{O}$ . Its degeneracy  $d_k$  on the other hand corresponds to the total number of the possible different row origins<sup>8</sup> of theses added boxes in the YD  $\psi$ . In our interpretation of the tensor product decomposition as associating fillings to the YD  $\psi$  to get a SSYT, the  $n_i$  in the  $N$ -tuple  $\beta$  counts the number of times the number  $i$  appears in the SSYT. As a result the  $N$ -tuple  $\beta$  is precisely a filling, see appendix-A. So, the degeneracy  $d_k$  is bounded from above<sup>9</sup> by the number of different SSYT $\psi$  that are associated to the same filling  $\beta$ . Such a number is called a Kostka number and is denoted by  $K_{\psi, \beta}$ , see appendix-A. Actually there is a precise relationship between Kostka numbers and degeneracies of YD appearing in the decomposition of a tensor product, see for example [32]. However, we will not need this exact relation in what we are trying to do in this paper.

Before continuing let us clarify a point that will play an important role in extracting the leading behavior of  $d_k$  in the following. We concluded that we can repackage the information about the tensor product  $\mathcal{O} \otimes \psi$  in terms of SSYT $\psi$  where the labels take values in  $\mathcal{L} = \{1, 2, \dots, N\}$ . We mentioned above that the number of SSYT $\psi$  with a certain filling  $\beta$  is given by the Kostka numbers  $K_{\psi, \beta}$ , however this is not completely exact. Essentially the Kostka number  $K_{\psi, \beta}$  encodes only the information about  $\psi$  and  $\beta$  being partitions of the same integer  $h$ , but not the information about the range of labels, see for example [33]. Remember that  $\beta$  is a collection of positive integers  $\beta_i$ , some of them can be zero, which are ordered according to the index  $i$ . We know on the other hand that a partition of an integer is a collection of strictly positive ordered integer that sum to that integer. So we seem to have a mismatch between the number of SSYT of interest to us and Kostka numbers: we can have zero entries in the filling  $\beta$  as well as ordering that depends on the index of the entry  $\beta_i$  and not its value. We will discuss the issue of the zeros here and leave the ordering issue to appendix-E. We will also discuss some further useful properties of Kostka numbers in the same appendix-E.

Remember that in order for a YT to be SSYT, the labeling should have a certain ordering along the rows and the columns of the associated YD (see appendix-A). This means that if  $\beta$  has some zero entries, the number of SSYT $\psi$  is the same as the one for a filling  $\tilde{\beta}$  which is constructed from  $\beta$  by omitting the zero terms, then relabeling its remaining entries  $\tilde{\beta}_i$  keeping the order of their index untouched. Take as an example  $\beta = (\beta_1, 0, 0, \beta_4, \beta_5)$ , then  $\tilde{\beta} = (\tilde{\beta}_1, \tilde{\beta}_2, \tilde{\beta}_3)$  such that  $\tilde{\beta}_1 = \beta_1$ ,  $\tilde{\beta}_2 = \beta_4$  and  $\tilde{\beta}_3 = \beta_5$ .

<sup>8</sup>Remember that boxes in the same row are treated symmetrically.

<sup>9</sup>Since we are not taking into account the modifications due to the YD and antisymmetry rules.

The reason that the number of SSYT $\alpha$  for both fillings is the same is that we did not alter the ordering of the index  $i$ , so the conditions on the order along the rows and column of the YT remain intact. Notice that  $\tilde{\beta}_i$  has the usual interpretation that it counts how many times the index  $i$  appears in the labeling of the YD.

#### 4.2 The tensor product $\mathcal{O} \otimes \psi$ : I-The case $h \ll N$

We are ready to tackle our main question of estimating the maximal possible degeneracy of a YD in the decomposition of  $\mathcal{O} \otimes \psi$  through the corresponding Kostka number. Our idea is to use  $\dim_N \psi$  since it is related to the number of possible SSYT $\alpha$  and hence to the Kostka numbers. Remember that  $\dim_N \psi$  is the number of all possible SSYT $\alpha$  with labels among the set  $\{1, 2, \dots, N\}$ . This number has two contributions: The first one is the number of possible choices of labels, and the second one is the associated Kostka number  $K_{\alpha, \beta}$ . Using that  $K_{\alpha, \beta} = 0$  if  $|\beta| < n$ , where  $n$  is the number of rows of  $\psi$  and  $|\beta|$  is the number of nonzero entries of the filling  $\beta$  (see appendix-E), it is easy to see that the dimension of  $\psi$  as a representation of  $SU(N)$  written in terms of Kostka numbers reads:

$$\dim_N \psi = \sum_{\beta; n \leq |\beta| \leq \text{Min}\{N, h\}} C_N^{|\beta|} K_{\psi, \beta}, \quad (4.4)$$

where  $n$  is the number of rows of  $\psi$ ,  $|\beta|$  stands for the number of nonzero elements in the filling  $\beta$ ,  $h$  is the number of boxes of  $\psi$ ,  $C_n^m$  is the usual binomial coefficient:

$$C_n^m = \frac{n!}{m!(n-m)!},$$

and  $K_{\psi, \beta}$  is the Kostka number associated to the filling  $\beta$ . The expression (4.4) above reflects the fact that when labeling the YD  $\psi$ , we first need to choose the labels to use, among all the possibilities  $\mathcal{L} = \{1, 2, \dots, N\}$  which gives rise to the binomial coefficient, then we need to decide on the possible multiplicities of these labels (i.e. choose a filling) each of which gives rise to its associated Kostka numbers.

Notice that not all fillings  $\beta$  are allowed in the sum (4.4) above as the corresponding Kostka number might vanish. Our approach will be to include all of these fillings  $\beta$  in the sum keeping in mind that some of the associated Kostka numbers can vanish. How many  $\beta$ 's are there? For a fixed  $|\beta|$ , the number of possible fillings  $\gamma$  such that  $|\gamma| = |\beta|$  is the same as the number of partitions of the integer  $h$  into  $|\beta|$  strictly positive integers  $\gamma_i$  that are ordered according to their index  $i$  and not their value. This is similar to the problem discussed in section-B.2 of appendix-B. The difference resides in the fact that we want to count partitions that do not have zero entries. It is easy to adapt the counting there to this case by putting from the start one ball in each box, reducing the number we want to partition to  $h - |\beta|$ . Hence we get the number:

$$\mathcal{N}_{|\beta|} = C_{h-1}^{|\beta|-1}. \quad (4.5)$$

It is clear that this is an over-estimate for the actual number of fillings  $\gamma$ , in the sense that not all the associated Kostka numbers are different from zero, but this will be enough for

our arguments in this and the next subsection. Using the estimate above, the total number of fillings  $\beta$  is bound from above by:

$$\mathcal{N}_\beta = \sum_{|\beta|=1}^h \mathcal{N}_{|\beta|} = 2^{h-1} . \quad (4.6)$$

It is an upper bound since we allowed for fillings  $\beta$  with  $|\beta| < n$  in this sum for which we know that their associated Kostka number vanishes. We have also allowed the range of summation to run all the way to  $h$  even if  $h > N$ . Notice that we are still missing the contribution of the binomial coefficient  $C_N^{|\beta|}$  to get the actual number of fillings, however we will not do so here and keep calling  $\mathcal{N}_\beta$  above the total number of fillings in an abuse of language.

As is clear from the form of the expression (4.4), our search for the “dominant” SSYT $\psi$  will depend on whether  $h$  or  $N$  is bigger. We will concentrate on the case  $h \ll N$  here and leave the cases  $h \sim N$  and  $h \gg N$  to the next subsection. Before continuing with our discussion, let us be more precise. First of all, what we really mean by comparing  $h$  and  $N$  is comparing the leading behavior of  $h$  with  $N$ . Hence,  $h = N - \sqrt{N}$  is part of the cases  $h \sim N$ . The next point we need to clarify is what class of probes we are interested in. Remember that we classified our probes into three families: generic, linear, and long probes, see subsection 2.2. Since we are interested in the leading behavior of the two point function  $\langle\langle\psi\psi\rangle\rangle_{\mathcal{O}}$ , given by equation (3.1), and as it is clear from the discussion in section 5 and the leading behavior of  $\dim_N \psi$  for the different families (section 2.2), as well as the fact that  $\dim_N \psi$  constitutes an upper bound on the total number of the YDs that appear in the tensor product decomposition of  $\mathcal{O} \otimes \psi$ , we will be only interested in the generic probes class. Remember that the YDs of this class are the ones with the number of rows  $n$  as well as the number of columns  $d$  are much smaller than  $N$  i.e.  $n, d \ll N$ .

The idea in the case  $h \ll N$  is to maximize both the binomial coefficient  $C_N^{|\beta|}$  and the Kostka number  $K_{\psi,\beta}$  independently. Notice that the sum in this case is over the range  $n \leq |\beta| \leq h$ . Hence, the maximum of the binomial coefficient  $C_N^{|\beta|}$  is reached for  $|\beta| = h$ . This is satisfied only for the filling  $\beta_0 = (1)^{10}$ , which gives rise to standard Young tableaux (SYTx). The reason we have only one filling  $\beta = (1)$  for  $|\beta| = h$  is that the number of fillings in a family with fixed  $|\beta|$  is the number of  $|\beta|$ -tuples  $(n_1, n_2, \dots, n_{|\beta|})$  such that:

$$\forall i; \quad n_i > 0, \quad \sum_{i=1}^{|\beta|} n_i = h .$$

It is clear from this condition that the only solution in the case  $|\beta| = h$  is  $n_i = 1, \forall i$ , and hence the claim that  $\beta_0 = (1)$  is the only filling such that  $|\beta| = h$ . What about the Kostka number  $K_{\psi,\beta}$ ? A moment thought reveals that this number is also maximized by the filling  $\beta_0 = (1)$ . This is because in this filling all the labels used to label the boxes of the YD  $\psi$  are different from each other. Combining this observation with the fact that for SSYT $\psi$ , the labels along the same column should be strictly increasing from top to bottom leads us

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<sup>10</sup>The notation  $\beta_0 = (1)$  means that all the entries in the filling  $\beta_0$  equal 1.

the aforementioned claim. As a result, the filling with the maximum contribution to the some in (4.4) is  $\beta_0 = (1)$ . Notice that maximum here does not mean that the contribution coming from the filling  $\beta_0 = (1)$  dominates over the other contributions to the sum (4.4). It will be nice to find out if there is a condition on the shape of the YD  $\psi$  such that this is true<sup>11</sup>. What is the value of the Kostka number  $K_{\psi, (1)}$ ? Remember that this is the same as the number of SYTx associated to the YD  $\psi$ , which in turn is the dimension of the YD  $\psi$  when seen as an irreducible representation of the permutation group  $S_h$  which is given by (see equation (2.13)):

$$K_{\alpha, (1)} = \dim_h \psi = \frac{h!}{\mathcal{H}_\psi} .$$

As a side note, one can get this formula by rescaling  $N$  as  $(\lambda N)$  in both the formulas (4.4) and (2.12) of  $\dim_N \psi$  without touching the YD  $\psi$ , then picking out the leading contribution to both of them in the limit  $\lambda \rightarrow \infty$  and equating them. This scaling argument may be helpful in answering the question raised previously about the condition of  $\psi$  so that the dominant contribution to (4.4) comes from SYTx.

Let us continue our investigation of the SYTx associated to  $\psi$  in the present situation i.e. the labels of the fillings of  $\psi$  are subsets of  $\mathcal{L} = \{1, 2, \dots, N\}$ . Their total number taking into account the contribution from the different choices of labels is given by  $\mathcal{N}_{SYT} = \mathcal{N}_{lab} \times \mathcal{N}_{perm}$ , where:

$$\log \mathcal{N}_{lab} = \log C_N^h \approx h \log \frac{N}{h} , \quad \log \mathcal{N}_{per} = \log \dim_h \psi \approx h \log \frac{h}{\psi_0} , \quad (4.7)$$

where  $\mathcal{N}_{lab}$  is the number of possible labelings and  $\mathcal{N}_{perm}$  the number of SYTx  $\psi$  given fixed labels. Notice that:

$$\log \dim_N \psi \approx \log \mathcal{N}_{lab} + \log \mathcal{N}_{perm} .$$

This is a very interesting observation as it suggests that we can restrict ourselves to the SYTx  $\psi$  when studying the SSYT  $\psi$  if we are only interested in leading order quantities. For example, the contribution from the other fillings to (4.4) is bounded from above by:

$$\delta \dim_N \psi \approx 2^{h-1} C_N^h K_{\psi, (1)} ,$$

which corrects the log of the contribution of the SYTx to  $\log \dim_N \psi$  by subleading terms as one expects based on the observation above.

Although we managed to get ample information about the SSYT  $\psi$  in the case  $h \ll N$ , we still lack the information we need:  $d_k$  the degeneracy of the YDs  $\varphi_k$  in (3.3) and their type. Using the discussion of the Kostka numbers above as a guide, and the map between SSYT  $\psi$  and  $\varphi_k$ , one expects that one can also take as representatives of the YDs  $\varphi_k$  appearing in the decomposition of  $\mathcal{O} \otimes \psi$  the ones that we get by adding at most one box to each row of  $\mathcal{O}$ . These are the “duals” of the SYTx  $\psi$  according to our map between SSYT and YDs  $\varphi_k$ , and we will denote them from now on by  $\varphi_k^0$ . The reasons that we

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<sup>11</sup>I am very grateful to Robert de Mello Koch for pointing out a mistake in an argument related to this point in a previous version of this paper.



can do so can be summarized as follows. First of all, the leading terms of log of their degeneracy  $d_k^0$  and their number  $\mathcal{N}_0$  which are given by:

$$\log d_k^0 \approx h \log \frac{h}{\psi_0} , \quad \log \mathcal{N}_0 \approx h \log \frac{N}{h} , \quad (4.8)$$

do not change once we take into account the YD and the antisymmetry rules. See section F.1 of appendix-F for more details. Secondly, the leading term of the log of  $\langle \varphi_k \varphi_k \rangle_{\mathcal{O}}$ , that appears in the explicit form (3.4) of the two point function (3.1), is the same for almost all the YDs  $\varphi_k$ , see section 5.1 for more details. Lastly, the degeneracies  $d_k$  are maximized for this kind of YDs. This is because  $d_k$  are related to Kostka numbers and these YDs are associated to the filling  $\beta = (1)$  which maximizes the Kostka number. We will come back to these issues and others when we discuss the full two point function of these probes in subsection 5.1.1.

#### 4.3 The tensor product $\mathcal{O} \otimes \psi$ : II-The cases $h \sim N$ and $h \ll N$

As previously mentioned, when studying the decomposition of the tensor product  $\mathcal{O} \otimes \psi$ , we are solely interested in the generic probes. Let us first look for a lower bound for the maximum possible value of  $K_{\alpha, \beta}$  using equation (4.4). Since it is the maximum, it is bigger or equal to the average value of  $K_{\psi, \beta}$  in this equation. To calculate this average we need to know the total number of terms in (4.4). Using the estimate (4.5) an over-estimate of the total number of terms is given by:

$$\mathcal{N}_{tot} = \sum_{k=0}^N C_N^k C_{h-1}^{k-1} < 2^{N+h} .$$

The combination of this estimate together with equation (4.4), the fact that  $\log \dim_N \psi \sim h \log N$  (see subsection 2.2), gives the following approximate value for  $K_{\alpha}^{max}$  the maximum value for the Kostka numbers:

$$\log K_{\psi}^{max} \approx \log \dim \psi \sim h \log N . \quad (4.9)$$

This means that there are some SSYT $\psi^*$  whose Kostka number is big enough to give rise to the leading term in  $\dim_N \psi$ . What are the fillings of such SSYT $\psi^*$ ? According to our discussion in appendix-E, the fillings  $\beta^*$  are such that all the labels  $1, 2, \dots, N$  are present with almost equal frequency.

Based on this results, one expects that there are few<sup>12</sup> YD  $\varphi_k^*$  in the decomposition (3.3) whose degeneracy  $d_k$  has the leading behavior:

$$\log d_k \approx \log \dim_N \psi \approx h \log \frac{N}{\psi_0} . \quad (4.10)$$

Once again, we need to check the effect of taking into account the YD and the antisymmetry rules. We will discuss the inclusion of the YD rule here as it is relatively easier to discuss

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<sup>12</sup>Few in the sense that the leading behavior of the log of their number is subleading with respect to  $h \log N$ .

than the inclusion of the other rule, the antisymmetry rule, which will be dealt with in section F.2 of appendix-F.

The fillings of these special SSYT $\times$   $\psi^*$  include all the numbers from 1 to  $N$ . So, the potential associated YD  $\varphi_k^*$  will be obtained by adding boxes to all the rows of  $\mathcal{O}$ . The YD rule then implies that the length of the new rows should be decreasing from top to bottom. The invariance of the Kostka numbers  $K_{\psi, \beta}$  under the reshuffling of the entries  $\beta_i$  of the filling  $\beta$  comes to our rescue, see appendix-E. Using this invariance, we choose the ordered filling  $\beta^*$ :  $\beta_1 \geq \beta_2 \dots \geq \beta_N$  to correspond to one of these possible  $\varphi_k^*$ . At this point one might be tempted to declare victory and conclude that taking into account the YD rule does not change the results obtained through the Kostka number means. However, one should remember that the YD rule is not limited to the final  $\varphi_k$ , but it is enforced after the addition of each box to the YD  $\mathcal{O}$ . Although we could not come up with a satisfactory argument to why the final conclusion will not change even after fully taking into account the YD rule, it does not seem to be that crazy to conjecture that this is the case. A piece of evidence has to do with the entries  $\beta_i^*$  of the filling  $\beta^*$ . Remember that for maximum Kostka numbers in our case of interest we have (see appendix-E for details):

$$\forall i \in \{1, 2, \dots, N\} ; \quad \beta_i^* \approx \frac{h}{N} .$$

Using that for our probes  $h \ll N^2$ , we conclude that  $\beta_i^* \ll N$ . This implies that it is more probable for boxes to end up in different rows, than in the same row. So, one can think of the cases where the YD rule is violated as organized constructions in contrast to the cases where it is satisfied, which can be thought of as random constructions. As a result, one can conclude that the number of cases where the YD rule is violated is subleading with respect to the total number of possible  $\varphi_k^*$ . This conclusion is in conformation with the assumption that the leading term of  $\log \langle \langle \psi \psi \rangle \rangle_{\mathcal{O}}$  is continuous, see subsection 5.1 for more details.

The inclusion of the antisymmetry rule is more involved and is left to section-F.2 of appendix-F as already advertised to. The final conclusion is that in the decomposition of the tensor product  $\mathcal{O} \otimes \psi$ , for  $\psi$  a generic probe and  $h \sim N$  or  $h \gg N$ , there are special YDs  $\varphi_k^*$  whose degeneracy  $d_k^*$  reproduces the leading behavior of  $\dim_N \psi$ , which is given in (4.10). Restricting ourselves to one of these YDs will give us the leading order of the log of the two point function (3.1) that we are after. For more details see subsection 5.1.2.

## 5. The full two point function and its universality

After we managed to gather the information we need on the decomposition of the tensor product  $\mathcal{O} \otimes \psi$ , we are almost ready to answer our main question of this paper: evaluate the leading term of the two point function (3.1) whose explicit expression is given by (3.4). The only remaining step is to evaluate the quantity  $\langle \varphi_k \varphi_k \rangle_{\mathcal{O}}$ . Our aim in this section is to evaluate the latter quantity, then get the leading term of the log of the full two point function (3.1). The discussion will depend on the type of the probe as well as the scaling of  $h$  with  $N$ . We will start by discussing the involved case of the generic probes, then move

on to the simplest cases of the linear and long probes. But before that, let us first deal with the general expression of  $\langle \varphi_k \varphi_k \rangle_{\mathcal{O}}$ .

To deal with the quantity  $\langle \varphi_k \varphi_k \rangle_{\mathcal{O}}$ , we need to introduce some notations that we will be using throughout this section. We will also remind ourselves of old conventions for convenience.  $\mathcal{O}_i$  is the length of the row  $i$  of the background YD  $\mathcal{O}$ . For almost all the values of  $i$ ,  $\mathcal{O}_i \sim N$ . Remember that  $i$  runs from 1 to  $N$ .  $d$  is the number of columns of  $\psi$ ,  $n$  is the number of its rows,  $h$  is the total number of its boxes,  $\psi_0 = \text{Max}\{n, d\}$ .  $\delta_i^{(k)}$  denotes the number of added boxes to row  $i$  of  $\mathcal{O}$  to form the YD  $\varphi_k$ , and  $n_\delta$  stands for the total number of non-zero  $\delta_i^{(k)}$ . The ordering rule implies that:

$$n_\delta \geq n, \quad \forall k, \forall i; \quad \delta_i^{(k)} \leq d. \quad (5.1)$$

These inequalities are also a straightforward implications of the properties of the SSYT labeling in conformation with the equivalence between the ordering rule and the SSYT $\psi$ . These inequalities will play an important role in the following.

Our starting point is equation (3.7) adapted to the present case:

$$\begin{aligned} \log \langle \varphi_k \varphi_k \rangle_{\mathcal{O}} \approx & -h + \sum_{i=1}^N (N + \mathcal{O}_i + \delta_i^{(k)} - i) \log(N + \mathcal{O}_i + \delta_i^{(k)} - i) \\ & - \sum_{i=1}^N (N + \mathcal{O}_i - i) \log(N + \mathcal{O}_i - i), \end{aligned} \quad (5.2)$$

To perform the sum above, we need to specify which probe and regime we are in. This will be one of our tasks in the remaining of this section.

### 5.1 The generic probes

Remember that for this class of probes we have  $d \ll N$ . This implies, according to (5.1), that  $\delta_i^{(k)} \ll N$ . Using that for almost all  $i$ 's  $\mathcal{O}_i \sim N$ , we can expand the log term in equation (5.2) and perform the sum to get:

$$\log \langle \varphi_k \varphi_k \rangle_{\mathcal{O}} \approx -h + \sum_{i=1}^N \delta_i^{(k)} \log(N + \mathcal{O}_i - i) \approx h \log N. \quad (5.3)$$

A legitimate objection to the use of this result in the expression (3.4) is that the rows where  $\mathcal{O}_i \ll N$  might spoil the final result. An in depth discussion of this point depends on the type of dominant YDs  $\varphi_k$ , which in turn depends on whether  $h \ll N$ , or ( $h \sim N$ ,  $h \gg N$ ). Since the leading term of the degeneracy  $d_k$  depends also on this splitting in the regimes of  $h$ , we will specialize below to these two distinct cases for the evaluation of the full two point function (3.1).

#### 5.1.1 The $h \ll N$ case

We concluded at the end of subsection 4.2 that in this case a good set of representative YDs  $\varphi_k^0$  are the ones resulting from adding at most one box to each row of  $\mathcal{O}$ . The number of

these YDs  $\mathcal{N}_0$  as well as their degeneracy have the leading terms that are given in equation (4.8), reproduced here for convenience:

$$\log d_k^0 \approx h \log \frac{h}{\psi_0} , \quad \log \mathcal{N}_0 \approx h \log \frac{N}{h} . \quad (5.4)$$

We need to settle the question of the usage of the expression (5.3) for all  $\langle \varphi_k \varphi_k \rangle_{\mathcal{O}}$  in the expression (3.4). The issue as pointed out earlier is that the behavior  $\mathcal{O}_i \sim N$  used in deriving this expression is not valid always. Although the ratio of the number of such rows in  $\mathcal{O}$  to the total number of rows  $N$  tends to zero in the large  $N$  limit, which is the limit we are interested in, one still should check that things do not go astray. First of all, notice that we only need to worry about these rows in the case where the number of boxes added to these rows is a finite fraction of the total number of boxes  $h$ . This is because we are interested in leading order terms. From the properties of the background YDs  $\mathcal{O}$ , see section 2.1, one easily infers that these problematic rows sit at the tail of the YD  $\mathcal{O}$ . Let us denote their number by  $m$ ,  $m \ll N$  but can be as big as we want. The probability that a box from  $\psi$  ends up in one of these rows is  $p = (m/N)$ . Let us suppose that  $\phi$  is a YD among the dominant  $\varphi_k$  YDs where  $(\kappa h)$  boxes from  $\psi$  are added to some rows among these  $m$  rows. Then the probability of finding such a YD among all the available YDs  $\varphi_k$  is bounded from above by:

$$\mathcal{P} \leq C_h^{\kappa h} p^{\kappa h} (1-p)^{(1-\kappa)h} \longrightarrow 0 , \quad \text{for } N \longrightarrow \infty ,$$

as one expects. This is an upper limit since we did not take into account that we still have a further contribution coming from the ratio between the total number of possibilities to distribute  $h$  boxes on  $N$  rows and the number of possibilities to distribute  $(\kappa h)$  boxes on  $m$  rows and the remaining boxes on  $(N-m)$  rows. This contribution reduces the probability further, but its effect is small, hence can be neglected.

As a result, when evaluating the contribution of the YDs  $\varphi_k^0$  to the leading term of the two point function (3.1) using the explicit expression (3.4), we can safely use (5.3) for  $\langle \varphi_k^0 \varphi_k^0 \rangle_{\mathcal{O}}$ , the degeneracy  $d_k^0$  as well as the number  $\mathcal{N}_0$  given by equation (5.4). At the end we get the following leading behavior of this contribution

$$\log \langle \langle \psi \psi \rangle \rangle_{\mathcal{O}}^0 \approx h \log N + 2h \log \frac{h}{\psi_0} + h \log \frac{N}{h} \approx h \log \left[ h \left( \frac{N}{\psi_0} \right)^2 \right] .$$

We need to worry about possible corrections to this leading behavior due to the other YDs. For this we need to remember that the degeneracies  $d_k$  are maximized by  $d_k^0$ , and that the leading term of the maximum of  $\langle \varphi_k \varphi_k \rangle_{\mathcal{O}}$  is the value used for  $\langle \varphi_k^0 \varphi_k^0 \rangle_{\mathcal{O}}$ . The final piece of information that we need is that the number of all possible YDs  $\varphi_k$  is bounded from above by:

$$\overline{\mathcal{N}} = C_N^h 2^h \approx 2^h \mathcal{N}_0 ,$$

see subsection 4.2 for more details. Using all these information, it is easy to conclude that the correction to the log of the full two point function  $\langle \langle \psi \psi \rangle \rangle_{\mathcal{O}}$  when we include all the other YDs  $\varphi_k$  is subleading. Hence, the log of the two point function for a generic probe

with  $h \ll N$  in the background of a typical state  $\mathcal{O}$  of the superstar ensemble has the following leading term:

$$\log \langle \langle \psi \psi \rangle \rangle_{\mathcal{O}} \approx h \log N + 2h \log \frac{h}{\psi_0} + h \log \frac{N}{h} \approx h \log \left[ h \left( \frac{N}{\psi_0} \right)^2 \right]. \quad (5.5)$$

This expression should be understood as follows. We should only take into account the leading behavior of the terms inside the log, whereas for  $h$  outside the log, we keep only its leading term. Notice the huge difference between this leading term and the corresponding one in the case of the backgrounds discussed in subsection 3.2, given by equation (3.8). So this class of probes  $\psi$  probe deep into the geometry, which allows us to distinguish between typical states and a random one. Notice that this distinction survives even in the case of near typical states, see the conclusions for more details.

### 5.1.2 The $h \sim N$ and $h \gg N$ cases

In these cases there are few dominant YD  $\varphi_k^*$  that are the result of adding  $\delta_i^{(k)} \approx (h/N)$  boxes to each row of  $\mathcal{O}$ . Their degeneracy is such that its leading term is:

$$\log d_k^* \approx \log \dim_N \psi \approx h \log \frac{N}{\psi_0}. \quad (5.6)$$

See section 4.3 and appendix-F section F.2 for more details. The reason we can use expression (5.3) for  $\langle \varphi_k^* \varphi_k^* \rangle_{\mathcal{O}}$  is much easier to understand in the present situation. Due to the way we build the dominant YDs  $\varphi_k^*$ , one easily concludes that the number of boxes added to the  $m$  rows whose length  $\mathcal{O}_i \ll N$  is less than:

$$\frac{h}{N} m \log m \sim \frac{m}{N} h \log N \ll h \log N,$$

which implies that the leading order of  $\langle \varphi_k \varphi_k \rangle_{\mathcal{O}}$  is the same as in the equation (5.3).

Now using the fact that  $\dim_N \psi$  is an upper bound on the total number of YDs  $\varphi_k$ , the leading term of the degeneracy  $d_k^*$  of the dominant YDs  $\varphi_k^*$ , as well as the leading behavior of  $\langle \varphi_k^* \varphi_k^* \rangle_{\mathcal{O}}$ , we find the following leading term of the two point function (3.1):

$$\log \langle \langle \psi \psi \rangle \rangle_{\mathcal{O}} \approx h \log N + 2h \log \frac{N}{\psi_0} \approx h \log \left[ N \left( \frac{N}{\psi_0} \right)^2 \right]. \quad (5.7)$$

Once again, we only care about scaling with  $N$  for what is inside the log, and for  $h$  outside the log we only keep its leading term in this expression. Using the same ideas as in the previous case, we can show that the inclusion of the other YDs  $\varphi_k$  adds only subleading corrections to this formula. It is also the case here that this leading term is different than the corresponding one discussed in subsection 3.2 and given by equation (3.8). Combining this observation with a similar one in the  $h \ll N$  regime, we conclude that the generic probes are efficient in distinguishing typical states from other backgrounds as well as other states in the superstar ensemble. For more details see the conclusions.

Another remark worth mentioning has to do with the analytic properties of the leading behavior of the log of the two point function of the generic probes. Comparing the expressions (5.5) and (5.7), it is easy to see that the leading term of  $\log \langle \langle \psi \psi \rangle \rangle_{\mathcal{O}}$  is continuous

for all values of  $h$ , but not differentiable at  $h \sim N$ . The continuity of the leading term is a result of the continuity of the two point function since subleading terms remain subleading when we change  $h$ . This continuity gives a further support of our conjecture that the leading term of the degeneracies  $d_k^*$  is given by the leading term of the associated Kostka number. See section 4.3 as well as appendix-F section F.2 for more details. On the other hand, the non-differentiability of  $\log \langle \psi \psi \rangle_{\mathcal{O}}$  happens at the point  $h \sim N$ . This is a point since the only difference between the expressions (5.5) and (5.7) resides in the term inside the log, and for such terms we only keep their scaling with  $N$  and not the full leading term. So we see a sign of a phase transition when the number of boxes  $h$  is of order  $N$ . This can be attributed to the fact that the gravity dual of the YD  $\psi$  with  $h \ll N$  is best thought of as gravitons, however when  $h$  becomes comparable to  $N$ , the duals are either giant or dual-giant gravitons [34, 35].

## 5.2 The linear probes

The discussion of this case is easier and this has to do with the fact that  $\log \dim_N \psi \sim h$  in this class of probes. Let us first look for  $\varphi_k$  that maximizes the expression (5.2). It is easy to see that this expressions reaches its maximum when the quantities  $\delta_i^{(k)}$  are maximized and the range of  $i$  is restricted to lie in the upper rows. A the solution to these two requirements is  $\delta_i = \psi_i$ . The associated YD  $\varphi_0$  always exists. There are other YDs that are a slight modification of this one, but they do not always exist and furthermore they do not change the conclusions to be reached here. Hence, we will just proceed with the YD  $\varphi_0$ . We start from the expression (5.2). To evaluate the sum, we need to distinguish between the two cases  $d \sim N$  and  $n \sim N$ . The first case is the easiest one. In this case we have  $n \ll N$ , hence we can neglect the correction coming from the presence of  $i$  in the log terms in the expression (5.2). Next, we use the fact that  $\mathcal{O}_i \sim \psi_i \sim N$  to find the following leading term:

$$\log \langle \varphi_0 \varphi_0 \rangle_{\mathcal{O}} \approx h \log N .$$

What about the second case  $n \sim N$ . In this case we have  $\psi_i \ll N$ . Taking advantage once again of the fact that most of the contribution to the sum in (5.2) will come from terms with  $\mathcal{O}_i \sim N$  and  $(N - i) \sim N$ , we get:

$$\log \langle \varphi_0 \varphi_0 \rangle_{\mathcal{O}} \approx h \log N ,$$

once again. Notice that here we used that  $\delta_i \ll N$  to expand the first log term. So, all in all, the largest possible value of  $\langle \varphi_k \varphi_k \rangle_{\mathcal{O}}$  in the present case has the following leading log term:

$$\log \langle \varphi_0 \varphi_0 \rangle_{\mathcal{O}} \approx h \log N . \tag{5.8}$$

After we found the maximum of  $\langle \varphi_k \varphi_k \rangle_{\mathcal{O}}$ , we need now to get the leading term of the expression (3.4). This is easily done taking advantage of the fact that  $\mathcal{N}$  the total number of YDs  $\varphi_k$  is much smaller than  $\langle \varphi_0 \varphi_0 \rangle_{\mathcal{O}}$ . The idea is to look for a maximum and a minimum of the sum in (3.4). A more than enough bounds are:

$$\langle \varphi_0 \varphi_0 \rangle_{\mathcal{O}} \leq \langle \psi \psi \rangle_{\mathcal{O}} \leq \mathcal{N} \times \langle \varphi_0 \varphi_0 \rangle_{\mathcal{O}} .$$

Now using that:

$$\log \mathcal{N} \leq \log \dim_N \psi \sim h \ll \langle \varphi_0 \varphi_0 \rangle_{\mathcal{O}} ,$$

we find:

$$\log \langle \langle \psi \psi \rangle \rangle_{\mathcal{O}} \approx h \log N . \quad (5.9)$$

Two observations are worth mentioning at this point. First of all, the linear probes are very bad in probing the states of the superstar ensemble if we restrict to leading order. This turns out to be case for the long probes as well as we will see below. This is because we get the same leading behavior as in the case of the background  $\mathcal{Q}$  discussed in subsection 3.2, see equation (3.9), even though the latter is not part of the superstar ensemble. This is because the linear probes probe the outer region of the geometry which is pretty much fixed by its asymptotic  $\text{AdS}_5 \times \text{S}^5$  geometry. The second observation is that the expressions (5.5), (5.7), and the expression (5.9) can be continuously connected to each other. However, due to non-differentiability at the “meeting” point  $\psi_0 \sim N$ , it seems that each class of probes have a different gravity duals. It could also be that part of this non-differentiability is associated to the fact that the probes are probing different regions of the background geometry.

### 5.3 The long probes

For these probes, we will use the same idea as in the linear probes case. This is because the total number of the YDs  $\varphi_k$  is “very small”. Actually, we can even get the first subleading term in this case. We start by maximizing  $\langle \varphi_k \varphi_k \rangle_{\mathcal{O}}$ . Once again, the YD  $\varphi_0$  for which  $\delta_i^{(0)} = \psi_i$  will do the job. Next, we need to evaluate the sum in expression (5.2). To do so, remember that in this class of probes  $\psi_i \sim d \gg N$  and  $n \ll N$ . It is straightforward to find the following leading term:

$$\log \langle \varphi_0 \varphi_0 \rangle_{\mathcal{O}} \approx h \log d - (\gamma + 1) h , \quad (5.10)$$

where  $\gamma$  is an  $N$ -independent constant that depends on the actual shape of the probe  $\psi$  and is given by:

$$\gamma = \frac{1}{h} \sum_{i=1}^n \psi_i \log \frac{d}{\psi_i} . \quad (5.11)$$

To get the leading term of the two point function (3.4), we use precisely the same trick as in the linear probes case. Remember that in this class of probes the total number of YDs  $\varphi_k$  satisfies the inequalities:

$$\log \mathcal{N} \leq \log \dim_N \psi \approx n N \log(d/N) \ll h \ll \log \langle \varphi_0 \varphi_0 \rangle_{\mathcal{O}} .$$

Using these in equalities, we find the following leading term:

$$\log \langle \langle \psi \psi \rangle \rangle_{\mathcal{O}} \approx h \log d - (\gamma + 1) h , \quad (5.12)$$

where  $\gamma$  is given by equation (5.11) above. This leading order is the same as the corresponding one for the background  $\mathcal{Q}$  given by equation (3.10) in subsection 3.2. This once

again confirms our intuition that long and linear probes are very bad in distinguishing the details of the background geometry as they probe regions outside the core, and hence have the same response to all backgrounds with the same asymptotic.

Looking at the expression (5.9) for the linear probes in the case  $d \sim N$  and the leading term of the expression (5.12) above, it is easy to see that we can unify them in a single simple expression that reads:

$$\log \langle \langle \psi \psi \rangle \rangle_{\mathcal{O}} \approx h \log d , \quad (5.13)$$

which has no explicit dependence on  $N$ . This suggests that for these type of probes it might be better to see them as fields in an  $U(d)$  theory rather than an  $U(N)$  theory. We will come to this point in the conclusions.

Notice that the leading terms of the log of the two point function  $\langle \langle \psi \psi \rangle \rangle_{\mathcal{O}}$  for the long probes and the linear probes for  $d \sim N$  can be connected continuously. Given that we can get the first subleading term in the case of log probes, one is tempted to use continuity to derive the corresponding first subleading term in the case of linear probes. Unfortunately, we cannot do so as some terms that are further subleading for the long probes, become comparable to  $h$  when we take the limit  $d \sim N$ .

## 6. Discussion and conclusions

The fuzzball proposal for black holes [5, 6, 7] has a lot of potential to solve the black hole puzzles. Even though there are several tests of this proposal (see the reviews [7, 8, 9, 10, 11, 12, 13] and the references therein.), it is not fully clear yet whether it is right or wrong, or needs just a reformulation. In this paper we started a further test of this proposal in the case of the superstar of [20] and its conjectured ensemble the superstar ensemble that was introduced in [1]. We probed the different typical states of this ensemble using light half-BPS probes. Of course, to complete the test one needs to do the dual calculation in the superstar geometry. We hope to come back to this in the future.

Although what we did in this paper was half the needed job, the results obtained strongly suggest that the conjecture of [1] might survive. We find the following universal leading term of the two point function (3.1):

$$\log \langle \langle \psi \psi \rangle \rangle_{\mathcal{O}} \approx \alpha h \log N , \quad (6.1)$$

where  $h$  is the energy/conformal weight of the probe  $\psi$ ,  $N$  the flux of the background we are probing, and  $\alpha$  is a constant that depends heavily on the shape of the probe  $\psi$  and on a lesser level on the scaling of  $h$  with  $N$ , see equations (5.5), (5.7), (5.9), and (5.12). By the shape of the YD  $\psi$ , we mean which of the two numbers: the number of columns  $d$  or the number of rows  $n$  dominates in the large  $N$  limit. The dependence of  $\alpha$  on the scaling of  $h$  with  $N$  in the generic probe class can be understood on the gravity side as a phase transition from gravitons in the case  $h \ll N$  to (dual-)giant gravitons when we hit the  $h \sim N$  point [34, 35].

One can also understand intuitively the dependence of the two point function  $\langle \langle \psi \psi \rangle \rangle_{\mathcal{O}}$  on  $N^h$  as follows. One can think about each box as a free graviton in our background. The



only thing that the graviton will see is the mass  $N^2$  of the background. So one expects that the two point function of a single free graviton to be proportional to  $N^2$ . So, once we are dealing with  $h$  free gravitons, we should get something proportional to  $N^{2h}$ . However, this result should be modified since the more gravitons we have, the less free they become. This interaction between the gravitons should modify the two point function. Intuitively, one can think about turning on the interaction as effectively reducing the number of gravitons. Hence one expects the two point function to take the form  $N^{\alpha h}$ . What about the value of  $\alpha$ ? can it be understood from the dual bulk perspective?

Intuitively, one can relate the dependence of  $\alpha$  on the shape of the probe  $\psi$  to the effects of angular momentum. Remember that the presence of angular momentum gives rise to centrifugal forces that forces the particles to move away from the center. In the gravitons regime ( $h \ll N$ ) there is a competition between the gravitational attraction that is proportional to the energy of the gravitons  $h$  and the centrifugal force that is set by the angular momentum. The latter depends on the shape of the probe. So it is not that strange that  $\alpha$  will depend on the ratio of these two quantities. Things are much cleaner when we move into the (dual-)giant gravitons regime ( $h \sim N$ ) or the regime of a bound state of gravitons and (dual-)giant gravitons ( $h \gg N$ ). Take for example the case  $d \gg n$ . In this case the dual probe is a bound state of  $n$  dual-giants in the  $\text{AdS}_5$  part of the geometry with a very large angular momentum which is proportional to  $d$ . Since the radius of the dual-giant is proportional to its angular momentum [35], these dual-giants probe different regions of the background geometry depending on the value of their angular momentum  $d$ . One can use this fact to argue for the independence of the value of  $\alpha$  from  $h$  in this regime. In the opposite case  $n \gg d$ , one talks about the giant gravitons inside the  $S^5$  part of the geometry. In this case we have a bound state of  $d$  giants with angular momentum given by  $n$ . The dependence of  $\alpha$  on  $n$  (and its independence from  $h$ ) can be explained using the same arguments as in the dual-giant case. This is because the radius of the giant is also proportional to its angular momentum [34]. Notice that the invariance of our result, equations (5.5), (5.7), (5.9), under the exchange (rows  $\leftrightarrow$  columns) can be seen as a manifestation of the symmetry (particle  $\leftrightarrow$  hole) in the quantum hall description of the half-BPS sector of type-IIB string theory on asymptotically  $\text{AdS}_5 \times S^5$  [29, 30]. So everything seems to fall in place.

We cheated a little bit when we declared that  $\alpha$  did not depend on the background. Naively one would expect it to depend on the shape of the background. This is the case and this dependence is encoded in the tensor product decomposition of  $\mathcal{O} \otimes \psi$ , see section 4 and appendix-F. However, since we are dealing with typical states of an ensemble which we claim that it has a well defined effective gravity description [1], the leading order of  $\alpha$  should have a definite value irrespective of the typical state we pick from the ensemble. This is precisely what we find. Actually we find more. If we try to move a little bit out of typicality, the value of  $\alpha$  changes. This can be observed in the discussion of appendix-F. There when we discussed the possible modifications to  $\alpha$  when we change the background YD  $\mathcal{O}$ , we concluded that as far as the number of rows, after collapsing each set of equal length rows to just one row, remains of order  $N$ ,  $\alpha$  remains intact. This was only satisfied by typical states. We can actually construct YDs  $\mathcal{O}$  that are very close to typicality but

with a different value of  $\alpha$ . These are YDs  $\overline{\mathcal{O}}$  where the rows are grouped into  $N^{1-a}$  sets of rows of equal length where each set has  $N^a$  rows, or YDs that are close enough to these. It is clear from the discussion in appendix-F that even for very small  $a \ll 1$ , we still get a different value of  $\alpha$ . The deviation is proportional to  $a$ , and hence small for small  $\alpha$ . So, we can still talk about near typical states, but that does not rule out the fact that we still get a different value for  $\alpha$  for these states.

In relation to the discussion above, we have seen that long and linear probes are not very good in probing the background geometry when restricting to leading order. This is not that surprising since their duals, as argued before, are (dual-)giant gravitons that are probing the outer region of the background geometry since they have a very large angular momentum. Since this outer region is pretty much fixed by the asymptotic  $\text{AdS}_5 \times \text{S}^5$  and the background flux, we expect that almost all the states in the superstar ensemble to lead to the same two point function for these probes. Actually one expects more, one expects that this leading term should be the same for all backgrounds with the same asymptotics as the asymptotics of the superstar ensemble states. This is backed up by the results obtained for the states  $\mathcal{Q}$  introduced in subsection 3.2. So for these type of probes we need to include the subleading terms to see the difference. As one expects, we need to go to higher subleading terms for the long probes, whereas for the linear probes it seems that the first subleading term will be enough to see some differences emerging. However, to distinguish between different typical states, it seems that we need to go even higher in our “expansion”.

One observation that was made at the end of section 5.3 and deserves a further discussion has to do with YD  $\psi$  whose number of columns  $d \gg N$ . It was observed that the leading term of the log of two point function  $\langle\langle\psi\psi\rangle\rangle_{\mathcal{O}}$  given by equation (5.12) did not depend on  $N$  explicitly. Taking into account a similar formula for  $d \sim N$ , equation (5.9), it was mentioned that these long probes will have the same leading order if we were dealing with backgrounds with flux  $d$ . Which suggests that one can pretty much conjecture that these probes have a non-trivial backreaction on the background. This could be the case when interpreting these probes as  $d$  giants in  $\text{S}^5$ . However, the picture is more obscure on the dual-giants side. It could be very well the case that since the dual-giant are close to the boundary, one should be careful about interpreting calculations on the dual field theory, see for example [36]. On the other hand, if we would have naively used very heavy probes  $h \gg N^2$ , we will end up always with these kind of YDs, since the number of rows is bounded by  $N$ , we will safely conclude that one should be dealing with a different dual field theory since these probes are heavier than the background and their backreaction should be taken into account. This gives a nice explanation of the “absence” of states of conformal weight  $h \gg N^2$  in  $\mathcal{N} = 4$   $\text{SU}(N)$  super Yang-Mills theory, even though there are no good reason for that. It is just that these states belong to a different theory when using AdS/CFT duality.

Finally, there is a very important property that we did not manage to find a trace of it in this paper. This has to do with the fact that the effective dual geometry to the superstar ensemble is singular. In our analysis, the leading term of the two point function (3.1) was continuous across all the regimes of parameters of the probe  $\psi$ . Although it was not

differentiable in all points of this parameter space, this non-differentiability was attributed to phase transition from gravitons to giant gravitons as well as to the fact that the dual gravity probes were probing different regions of the bulk geometry. One questionable point in this parameter space is when the number of columns  $d$  moves from being of order  $N$  or less to the case where  $d \gg N$ . This is because the former probe inside the background geometry and the latter probe the outside regions. So, one is tempted to associate the non differentiability at the point  $d \sim N$  to this singularity. However, as argued above the long probes  $d \gg N$  are kind of weird and one needs to be very careful when interpreting results that one obtains when using them as probes. Modulo this point, it seems that if there is a signature of this singularity, we need to go beyond the leading term or study higher point functions. We hope that future studies will help in elucidating this point.

There are several things to be done to complete the discussion in this paper. The first one, which was mentioned already, is to do the dual calculation in the superstar background. Another direction of research has to do with the simple final result (6.1) that we got. Its simplicity begs for a better approach to derive it instead of our “tour-de-force” approach. Maybe a better use of the large  $N$  technology could help, especially the free fermion or matrix models language. A third open question has to do with beyond the leading term calculation. Unfortunately, the techniques developed here are limited to the leading order, and things get wild when going beyond except for long probes where we could calculate the first subleading term (5.12). Given that the long probes are the worst to use, this does not seem to be such an interesting progress. Maybe a different approach to the problem can help here. Actually, one of the original motivations to this work was to try to distinguish the typical states of the superstar ensemble. We found, as expected, that we need to go beyond the leading term for that. The question is then, at which subleading term can we start to distinguish some typical states from the others. According to the calculations in this paper, it seems that we might have some chance in the first subleading correction in the case of generic probes. However our technology that we used here comes short to this task. In the same line of thoughts, one wonders if there is a nice expansion parameter for  $\log \langle \langle \psi \psi \rangle \rangle_{\mathcal{O}}$ ? It seems that the general structure will be an alternation of terms that contain  $\log N$ , and terms without this log. It does seem also that for generic probes,  $(1/\psi_0)$  might be the parameter of expansion, however we could not come up with a nice reason to why this is the case. It is possible that this parameter will work only for the first few subleading terms since it reflects the separation of the (dual-)giants in the background geometry, and one would expect that at certain stage the flux of the background will kick in. This happens for the second subleading term in the case of the long probes as the parameter of expansion seems to be  $(N/d)$ . Actually, it is not even clear that there will be a nice expansion in the first place. Hopefully, further works tackling the same problem discussed in this paper using better technology will help shed light on these open questions and others.

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## A. Conventions and notation

In this appendix we will summarize our conventions and notation that is used throughout this paper.

**Young diagrams (YD)** Throughout this paper we will use YD as a short notation for a Young diagram and YDs for more than one. A YD is a collection of boxes, arranged in left-justified rows, with the row lengths weakly decreasing from top to bottom. For example:

$$\psi = \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array},$$

is a YD. We start enumerating rows from top to bottom and columns from left to right. Our main use of YDs in this paper is through their connection with irreducible representations of the unitary and permutation groups.

**The shape of a YD** is the collection of the lengths of the rows of the YD into an  $n$ -tuple of integers. For example the shape of the previous YD is  $\psi = (4, 3, 2, 2)$ . We will be using the same symbol to denote the YD as well as its shape.

**The hook lengths of a YD** A box in a YD is denoted by the pair  $(i, j)$  where  $i$  is the number of the row and  $j$  is the number of the column whose intersection is this box.  $h_{(i, j)}$  stands for the hook length associated to the box  $(i, j)$ . The hook length associated to a box is the number of boxes below and to the left of it plus one. As an example, we depict below a YT where its boxes are filled with their associated hook lengths  $h_{(i, j)}$ :

$$\begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|c|} \hline 7 & 6 & 2 & 1 \\ \hline 5 & 4 & 1 & \\ \hline 3 & 2 & & \\ \hline 2 & 1 & & \\ \hline \end{array}.$$

The product of all the hook lengths plays an important role in the formulation of the dimension of the  $SU(N)$  group or the permutation group representation given by this YD, see equations (2.12) and (2.13).

**Young tableau (YT)** Throughout this paper we will use YT as a short notation for a Young tableau and YT $x$  for more than one. A YT is a YD where each box is filled with a number<sup>13</sup>. The collection of these number is called a **filling**, and will be denoted by a

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<sup>13</sup>This is not the only possibility as we can use letters for example to label the boxes. However we will be mainly interested in labels that are strictly positive numbers

Greek letter e.g.  $\beta$ . We will use the word **labeling** of the YD to refer to all possible fillings  $\beta$ . It is more convenient to denote a filling  $\beta$  by  $\beta = (\beta_1, \beta_2, \dots, \beta_m)$ , where  $\beta_i$  denotes the number of times the integer  $i$  appears. We denote by  $|\beta|$  the **length** of the filling  $\beta$  which is the number of its non-zero entries  $\beta_i \neq 0$ . As an example, the following YTx:

$$\begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 2 \\ \hline 2 & 4 & 4 & \\ \hline 4 & 5 & & \\ \hline 5 & 7 & & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|c|} \hline 5 & 7 & 2 & 2 \\ \hline 4 & 4 & 4 & \\ \hline 2 & 5 & & \\ \hline 1 & 1 & & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|c|} \hline 5 & 7 & 2 & 5 \\ \hline 1 & 4 & 2 & \\ \hline 2 & 2 & & \\ \hline 4 & 4 & & \\ \hline \end{array}, \quad \dots$$

correspond to the YD of the previous example with the filling  $\beta = (2, 3, 0, 3, 2, 0, 1)$ . The length of this filling is  $|\beta| = 5$ .

**Standard Young tableau (SYT)** Throughout this paper we will use SYT as a short notation for a standard Young tableau, and SYTx for more than one. Let  $h$  be the number of boxes of a YD under consideration. A SYT is a YD whose boxes are filled with numbers  $1, 2, \dots, h$ , each one occurring once and such that the numbers on the same row are strictly increasing from left to right, and the numbers on the same column are strictly increasing from top to bottom. The following YTx depict some examples of SYT.

$$\begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 & \\ \hline 8 & 9 & & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|c|} \hline 1 & 3 & 4 & 5 \\ \hline 2 & 6 & 9 & \\ \hline 7 & 8 & & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|c|} \hline 1 & 4 & 9 & 7 \\ \hline 2 & 5 & 8 & \\ \hline 3 & 6 & & \\ \hline \end{array}, \quad \dots$$

The number of all SYTx associated to a YD  $\psi$  gives the dimension of the representation of the permutation group  $S_h$  given by this YD  $\psi$ , see for example [27].

**Semi-standard Young tableau (SSYT)** Throughout this paper we will use SSYT as a short notation for a semi-standard Young tableau and SSYTx for more than one. A SSYT is a YD whose boxes are filled with positive integers such that the numbers on the same row are weakly increasing from left to right and the numbers on the same column are strictly increasing from top to bottom. The following YTx are examples of SSYT.

$$\begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 2 \\ \hline 2 & 4 & 4 & \\ \hline 4 & 5 & & \\ \hline 5 & 7 & & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 7 \\ \hline 2 & 2 & 4 & \\ \hline 4 & 4 & & \\ \hline 5 & 5 & & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 5 \\ \hline 2 & 2 & 4 & \\ \hline 4 & 4 & & \\ \hline 5 & 7 & & \\ \hline \end{array} \quad \dots$$

The number of all possible SSYTx associated to a YD  $\psi$ , whose labels are integers running from 1 to  $N$  is the dimension of the representation of the group  $SU(N)$  given by this YD  $\psi$ , see for example [27].

**Kostka numbers** The Kostka number  $K_{\psi, \beta}$  is the number of SSYTx associated to the YD of shape  $\psi$  and the filling  $\beta$ .

**Background YD conventions** In this paper we will use the letter  $\mathcal{O}$  to denote the background under consideration, be it a state, a geometry, or a dual YD. We will denote by  $\mathcal{O}_i$  the length of the row  $i$  of the YD  $\mathcal{O}$ . We will reserve the notation  $\mathcal{O}_0$  to the limit shape YD of the ensemble under consideration. The energy/conformal dimension/number of boxes of the background will be denoted by  $\Delta$ , whereas the number of its columns will be denoted by  $D$ . For characteristics of these backgrounds see section 2.1.

**Probe YD conventions** In this paper we will use the letter  $\psi$  to denote the probe, be it a state, or a dual YD. Its energy/conformal dimension/number of boxes will be denoted by  $h$ .  $\psi_i$  will denote the length of the row  $i$ . We will also reserve the letter  $d$  to denote the number of its columns ( $d = \psi_1$ ), the letter  $n$  to denote the number of its rows,  $\psi_0 = \text{Max}\{n, d\}$ , and the letter  $\mathcal{H}$  to denote the product over all its hook lengths. See section 2.2 for characteristics of the probe YDs that will be used in this paper.

**The dimension of a YD** Two dimensions associated to the probe YD  $\psi$  will play an important role in this paper.  $\dim_N \psi$  will denote the dimension of  $\psi$  seen as an irreducible representation of the group  $\text{SU}(N)$ , whereas  $\dim_h \psi$  stands for its dimension seen as an irreducible representation of the permutation group  $S_h$ .

**Leading behavior vs leading term** Two notions that will be used extensively in the paper: leading behavior and leading term. By the leading behavior of a quantity  $A$  we mean its scaling behavior with  $N$ . We use the symbol “ $\sim$ ” to express the leading behavior. On the other hand, the leading term of a quantity  $A$  means the dominant term in an expansion of  $A$  in the large  $N$  limit. We will use the symbol “ $\approx$ ” to express the leading term. For example:

$$A = \sum_{k=1}^N k^2 = \frac{1}{6} N(N+1)(2N+1), \quad A \sim N^3, \quad A \approx \frac{1}{3} N^3.$$

## B. A collection of some useful formulas

Most of the results derived in this paper rely on formulas that will be discussed in this appendix. The first kind of formulas has to do with approximating sums of the form:

$$\mathcal{S}_\ell(h, a; n) = \sum_{k=0}^n (hk + a)^\ell \log(hk + a),$$

for  $n \gg 1$ , and  $\ell \leq 2$ , which will be the topic of the first section below. The second formula has to do with counting the possible ways to partition a positive integer  $n$  into  $m$  positive integers  $n_i$ ;  $i = 1, 2, \dots, m$ , where the order of the index  $i$  is important. We will be more precise about what are we really counting in the second section below. In the last section we will discuss some useful properties of polylogarithm function  $\text{Li}_n(x)$  that will be used in appendix-C.

## B.1 Sum approximation

In the following we will discuss an approximation to sums of the form:

$$\mathcal{S}_\ell(h, a; n) = \sum_{k=0}^n (h k + a)^\ell \log(h k + a), \quad (\text{B.1})$$

for  $n \gg 1$ . There are different approaches to deal with this kind of sums. We will use the Euler-Maclaurin approximation which is valid for all smooth real functions  $f(x)$ . We have (see [37] for example):

$$\sum_{k=0}^n f(h k + a) = \frac{1}{h} \int_a^b f(x) dx + \frac{f(a) + f(b)}{2} + \sum_{k=1}^p \frac{h^{2k-1}}{(2k)!} B_{2k} \left( f^{(2k-1)}(b) - f^{(2k-1)}(a) \right) + \mathcal{R}_h, \quad (\text{B.2})$$

where  $b = a + h n$ , and  $\mathcal{R}_h$  is the remainder and is given by:

$$\mathcal{R}_h = \frac{h^{2p+2}}{(2p+2)!} B_{2p+2} \sum_{k=0}^{n-1} f^{(2p+2)}(a + h k + \theta), \quad (\text{B.3})$$

where  $\theta$  is some constant between 0 and 1 i.e.  $0 < \theta < 1$ . It can also be shown that  $\mathcal{R}$  is bounded by:

$$|\mathcal{R}| \leq \frac{2\zeta(2p)}{(2\pi)^{2p}} \int_1^n |f^{(2p)}(x)| dx, \quad (\text{B.4})$$

where  $\zeta(t)$  is the Riemann zeta function. The constants  $B_n$  that appear in the relation (B.2) are the Bernoulli numbers with  $B_1 = 1/2$ . Their generating function is given by:

$$\sum_{m=0}^{\infty} B_m \frac{t^m}{m!} = \frac{t}{1 - e^{-t}} := \mathcal{B}(t). \quad (\text{B.5})$$

It is clear from the generating function above that if  $n \geq 1$  then  $B_{2n+1} = 0$ . The easiest way to see this is by using the observation:

$$\mathcal{B}(t) - \mathcal{B}(-t) = t.$$

To apply the Euler-Maclaurin formula to our sum (B.1), we need to calculate the integral as well as the derivatives of the function  $f(x) = x^\ell \log x$ . We have:

$$\begin{aligned} \int f(x) dx &= \frac{1}{\ell+1} x^{\ell+1} \left( \log x - \frac{1}{\ell+1} \right), \\ f^{(i)}(x) &= \frac{\ell!}{(\ell-i)!} x^{\ell-i} \left( \log x + \sum_{j=1}^i \left[ \frac{1}{\ell+1-j} \right] \right); \quad \text{for } i < \ell, \\ f^{(m)}(x) &= \ell! \log x + \dots \end{aligned}$$

The last ingredient we need is the values of the first few Bernoulli numbers. We have:

$$B_0 = 1, \quad B_1 = \frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}.$$

Putting everything together we get the following approximations to the first few sums:

$$\mathcal{S}_0(h, a; n) \simeq \frac{1}{h} \left[ \left( b + \frac{h}{2} \right) \log b - \left( a - \frac{h}{2} \right) \log a \right] - n, \quad (\text{B.6})$$

$$\mathcal{S}_1(h, a; n) \simeq \frac{1}{2h} \left[ \left( b(b+h) + \frac{h^2}{6} \right) \log b - \left( a(a-h) + \frac{h^2}{6} \right) \log a \right] - \frac{1}{4} n(b+a), \quad (\text{B.7})$$

$$\begin{aligned} \mathcal{S}_2(h, a; n) \simeq \frac{1}{3h} \left[ b(b+h) \left( b + \frac{h}{2} \right) \log b - a(a-h) \left( a - \frac{h}{2} \right) \log a \right] - \\ - \frac{1}{9} n(a^2 + b^2 + ab) + \frac{h^2}{12} n, \end{aligned} \quad (\text{B.8})$$

where we truncated the sums to the log order which is enough for our purposes.

In the remaining of this subsection, we will use the approximations above to evaluate a double sum that will appear frequently in appendix-D. This double sum is given by:

$$\log \mathcal{H}_0 = \sum_{i=0}^{n-1} \sum_{j=0}^{d-1} \log(\mathcal{N} + 1 + i + j), \quad (\text{B.9})$$

where  $\mathcal{N}$  is some positive integer. The idea is to use the special structure of the summand to change the double sum over  $i$  and  $j$  to a single one over  $k = i + j$ . For that we need to introduce the following quantities:

$$a = \text{Min} \{n, d\} \quad , \quad b = \text{Max} \{n, d\} .$$

Notice that  $a + b = n + d$ . It is easy to see that one gets:

$$\begin{aligned} \log \mathcal{H}_0 &= \sum_{k=1}^{a-1} k \log(\mathcal{N} + k) + a \sum_{k=a}^b \log(\mathcal{N} + k) + \sum_{k=b+1}^{a+b-1} (a+b-k) \log(\mathcal{N} + k) \\ &= H_{\mathcal{N}}(n+d) - H_{\mathcal{N}}(n) - H_{\mathcal{N}}(d), \end{aligned} \quad (\text{B.10})$$

where  $H_{\beta}(\alpha)$  is a short notation for the sum:

$$H_{\beta}(\alpha) = \sum_{k=1}^{\alpha-1} (\alpha - k) \log(\beta + k), \quad (\text{B.11})$$

It is a straightforward exercise to get an approximate expression for  $H_{\beta}(\alpha)$  using equations (B.6) and (B.7). We get:

$$\begin{aligned} H_{\beta}(\alpha) &= (\alpha + \beta) \sum_{k=0}^{\alpha-2} \log(\beta + 1 + k) - \sum_{k=0}^{\alpha-2} (\beta + 1 + k) \log(\beta + 1 + k) \\ &\simeq \frac{1}{2} \left[ (\alpha + \beta)^2 - \frac{1}{6} \right] \log(\alpha + \beta) - \frac{1}{2} \left[ \beta^2 + \alpha(2\beta + 1) - \frac{1}{6} \right] \log \beta - \frac{1}{4} \alpha(2\beta + 3\alpha). \end{aligned} \quad (\text{B.12})$$



Plugging in this result into the expression (B.10) of  $\log \mathcal{H}_0$  above, we find:

$$\begin{aligned} \log \mathcal{H}_0 \simeq & \frac{1}{2} \left( \mathcal{N} + n + d - \frac{1}{6} \right) \log(\mathcal{N} + n + d) + \frac{1}{2} \left( \mathcal{N} - \frac{1}{6} \right) \log \mathcal{N} - \\ & - \frac{1}{2} \left( \mathcal{N} + n - \frac{1}{6} \right) \log(\mathcal{N} + n) - \frac{1}{2} \left( \mathcal{N} + d - \frac{1}{6} \right) \log(\mathcal{N} + d) - \frac{3}{2} n d . \end{aligned} \quad (\text{B.13})$$

## B.2 A counting formula

We will need at different places in this paper to count the possible ways to partition a positive integer  $n$  into  $m$  positive integers  $n_i$ ;  $i = 1, 2, \dots, m$ . More precisely we are interested in counting the  $m$ -tuples  $(n_1, n_2, \dots, n_m)$  such that:

$$\forall i ; \quad n_i \geq 0 , \quad \sum_{i=1}^m n_i = n .$$

Notice that this is different, and much easier, than the usual notion of a partition of integers. In the latter case the order of the index  $i$  is not important i.e. we are counting sets of integers instead of  $m$ -tuples. On top of that in the usual counting of partitions, we require the integers  $n_i$  to be strictly positive.

The answer to our counting question is easy. It is the same as counting the number of ways to distribute  $n$  balls on  $m$  boxes. The answer to the latter question is given by:

$$\mathcal{P}_m(n) = C_{n+m-1}^n , \quad (\text{B.14})$$

where  $C_a^b$  is the binomial coefficient:

$$C_b^a = \frac{b!}{a! (b-a)!} .$$

## B.3 The polylogarithm function

In this section we will summarize some of the important properties of the polylogarithm functions  $\text{Li}_n(z)$ , that will be of importance to us in appendix-C. Since, in appendix-C, we will be dealing with quantities that are of the form  $e^{-\beta x - \lambda y}$ , where  $\beta, x, \lambda$  and  $y$  are all positive, we restrict the argument  $z$  of the function  $\text{Li}_n(z)$  to lay in the interval  $0 < z < 1$ . We have by definition:

$$\text{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n} ; \quad \forall n > 0 . \quad (\text{B.15})$$

It is easy to see, using the definition above that:

$$\int_0^z \frac{\text{Li}_n(y)}{y} dy = \text{Li}_{n+1}(z) ; \quad \text{for } 0 < z < 1 , \quad (\text{B.16})$$

$$z \partial_z \text{Li}_n(z) = \text{Li}_{n-1}(z) . \quad (\text{B.17})$$

One can use the second property to define polylog for negative  $n$ . We find:

$$\text{Li}_{(-n)}(z) = (z \partial_z)^n \frac{z}{1-z} ; \quad \forall n \geq 0 , \quad (\text{B.18})$$

where the case  $n = 0$  corresponds to the function that the operator  $(z \partial_z)^n$  acts on.

The last property we would like to discuss is the Taylor expansion of the polylogarithm function whose argument takes the special form  $z = z_0 e^\alpha$ , where  $\alpha \ll 1$ . It can be easily proved, using the defining equation of the polylogarithm (B.15) that:

$$\text{Li}_n(z_0 e^\alpha) - \text{Li}_n(z_0) \simeq \sum_{k=1}^n \frac{\alpha^k}{k!} \text{Li}_{n-k}(z_0) + \dots \quad (\text{B.19})$$

This expansion is more than enough for our purposes even though we truncated the sum to  $n$ .

### C. Thermodynamics of the superstar ensemble

In this appendix, we will evaluate some thermodynamical quantities of our superstar ensemble of background YDs, see section 2.1. Our first step is to evaluate the log of the partition function  $\mathcal{Z}$  defined in equation (2.3), using the approximation (B.2). We will check the validity of our approximation at the end. After that, we will move on to fix the values of the parameters  $\beta$  and  $\lambda$ . All the expressions derived below can be found in [1] with appropriate modifications.

Our starting point is the defining equation of  $\mathcal{Z}$ , equation (2.3), that we rewrite here for convenience.

$$\log \mathcal{Z} = - \sum_{j=1}^N \log(1 - p q^j) = \sum_{j=1}^N \text{Li}_1(p q^j) \quad , \quad (\text{C.1})$$

where  $\text{Li}_n(x)$  stands for the polylog function defined in (B.15). To evaluate the sum that appears in the expression of  $\log \mathcal{Z}$  above using the approximation (B.2), we need to take  $h = 1$ ,  $a = 1$ ,  $b = N$ , and  $f(x) = \text{Li}_1(p q^x)$ . We have:

$$\int_1^N \text{Li}_1(p q^x) dx = \frac{1}{\log q} \int_{pq}^{pq^N} \frac{\text{Li}_1(y)}{y} dy = \frac{1}{\log q} [\text{Li}_2(p q^N) - \text{Li}_2(p q)] \quad , \quad (\text{C.2})$$

where we used the change of variables  $y = p q^x$ , then the property (B.16) of the polylog functions since both  $q$  and  $p$  are positive and less than one, and  $N$  is a positive integer. Next, we need to evaluate the derivatives of  $f(x)$ . One can prove using the property (B.17) and the definition (B.18) that:

$$f^{(n)}(x) = (\log q)^n \text{Li}_{(1-n)}(p q^x) \quad . \quad (\text{C.3})$$

The proof proceeds by using once again the variable  $y = p q^x$ , then the chain rule to change the derivative with respect to  $x$  to:

$$\partial_x = (\log q) (y \partial_y) \quad .$$

Collecting everything, we find the following approximate expression for  $\log \mathcal{Z}$ :

$$\begin{aligned} \log \mathcal{Z} \simeq & \frac{1}{\log q} [\text{Li}_2(p q^N) - \text{Li}_2(p q)] + \frac{1}{2} [\text{Li}_1(p q^N) + \text{Li}_1(p q)] \\ & + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} (\log q)^{(2k-1)} [\text{Li}_{(2-2k)}(p q^N) - \text{Li}_{(2-2k)}(p q)] \quad , \end{aligned} \quad (\text{C.4})$$

Although we managed to evaluate  $\log \mathcal{Z}$ , we need to check the validity of our approximation. Luckily enough, our  $\beta = -\log q$  is very small [1, section 3.3]. In this regime we can safely neglect the sum over  $k$  in equation (C.4), to get the more manageable approximation:

$$\log \mathcal{Z} \simeq \frac{1}{\log q} [\text{Li}_2(p q^N) - \text{Li}_2(p q)] + \frac{1}{2} [\text{Li}_1(p q^N) + \text{Li}_1(p q)] , \quad (\text{C.5})$$

To evaluate  $\Delta$ , we plug this expression in the relation (2.4) to get:

$$\Delta \simeq \frac{1}{(\log q)^2} [\text{Li}_2(p) - \text{Li}_2(p q^N)] - \frac{1}{(\log q)} [\text{Li}_1(p) - N \text{Li}_1(p q^N)] + \frac{N}{2} \text{Li}_0(p q^N) , \quad (\text{C.6})$$

where we used the property (B.17). To make connection with a similar expression that appears in [1], we need to use the fact that  $\text{Li}_1(x) = -\log(1-x)$  and the identity:

$$\text{Li}_2(y) + \text{Li}_2(1-y) = \frac{\pi^2}{6} - \log y \log(1-y) .$$

The expression (C.5) above together with the relation (2.5), gives for  $D$  the following expression<sup>14</sup>:

$$D \simeq \frac{1}{\log q} [\log(1-p) - \log(1-p q^N)] . \quad (\text{C.7})$$

Observe that we can arrive at the same expressions as (C.6) and (C.7) by starting from (2.4) and (2.5), then performing the sum following the same steps as in evaluating  $\log \mathcal{Z}$ . The value of  $p$  can be already fixed using the constraint (C.7). We find:

$$p = \frac{1 - q^D}{1 - q^{D+N}} . \quad (\text{C.8})$$

We need also a way to fix  $\beta$ , however the expression (C.6) even after substituting  $p$  by its value is complicated and the solution depends on the regime of  $\beta$ . We will take a slight detour here to have an idea about the regime of  $\beta$ , then we will come back to its value later on. The idea is to use the constraint on the limit shape YD of this ensemble [1]. Remember that we want the limit shape curve to be a line. This because the superstar maximized the entropy. The latter is associated to moving around the outer boxes of the YDs. So, naively one would expect that the typical YDs of these ensemble should be closer to a triangular YD [1].

Following [1], we start by<sup>15</sup>:

$$y(x) = \sum_{i=x}^N \langle c_i \rangle ,$$

where  $x$  is the coordinate along the rows that increases from the top to the bottom of YDs, and  $y(x)$  is the length of the row  $x$ . Using the expression for  $\langle c_i \rangle$  that appears in (2.5), we find the following equation for the limit shape:

$$(1 - p q^N) q^y + p q^x = 1 .$$

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<sup>14</sup>It turns out that keeping only the leading term is enough for our purposes.

<sup>15</sup>We have a different expression for  $y(x)$  than the one in [1] since we have different conventions.

One can easily check that the box  $(N, 0)$  is part of this limit shape curve. Requiring on the other hand that  $(0, D)$  is also part of this limit shape curve leads to the same value of  $p$  as before, equation (C.8). Plugging this value into the expression above leads to the following limit shape curve equation:

$$(1 - q^N) q^y + (1 - q^D) q^x = 1 - q^{D+N} . \quad (\text{C.9})$$

To get a straight line out of this equation, we need to bring down  $x$  and  $y$  from the exponent. Since both  $x$  and  $y$  are of order  $N$ , we need  $(\beta N) \ll 1$  i.e.  $\beta \ll (1/N)$ . If this is satisfied, one gets for  $p$  the following approximate value:

$$p \approx \frac{D}{D + M} . \quad (\text{C.10})$$

The limit shape curve becomes in this case:

$$y \approx D \left( 1 - \frac{x}{N} \right) , \quad (\text{C.11})$$

which is a straight line.

To fix completely  $\beta$ , one can try to use the fact that  $(\beta N \ll 1)$ , expand the expression (C.6) appropriately using equation (B.19) and equate  $\Delta$  to  $(ND)/2$ . However, one gets exactly the latter equality independently of  $\beta$ . One could have anticipated this since the limit shape YD is a triangle and  $\langle \Delta \rangle$  should give to leading order the same number of boxes as in this YD, which is precisely  $(ND)/2$ , a value that is independent of  $\beta$ .

Our last card is to use the entropy  $S$  of this ensemble. We have:

$$S = \beta \Delta + \lambda D + \log \mathcal{Z} \approx -N \log(1 - p) - D \log p + \mathcal{O}(\log^2 q) \sim N . \quad (\text{C.12})$$

Notice that there is no linear term in  $\beta = \log q$ , so the superstar ensemble extremizes the entropy. It can be shown easily by calculating the coefficient in front of  $\log^2 q$ , that the superstar ensemble maximizes the entropy [1]. Using the relation between the inverse temperature  $\beta$ , the energy  $\Delta$  and the entropy  $S$ , we find [1]:

$$\beta = \frac{\partial S}{\partial \Delta} \sim \frac{1}{N} , \quad (\text{C.13})$$

where we used that  $\Delta \sim N^2$ . Finally, we should remember that in the expressions above we set the Plank constant to one  $\hbar = 1$  [1]. Once we restore it, we find  $N \log q \ll 1$  even though the scaling of  $\log q$  with  $N$  is  $\log q \sim 1/N$ .

A final observation is in order here. A careful treatment of this ensemble and its entropy leads to an agreement between the latter and the entropy of the superstar of [20]. This led [1] to conjecture that the superstar geometry should be seen as an effective description of this ensemble, hence the name. For more details and evidences see [1].

## D. A non-trivial simple family of Young diagrams

In the following we are going to evaluate the dimension of a family of YDs that we call from now on the optimum family, that is generic enough for our purposes and at the same time

simple enough to allow for exact evaluation of the leading term of the log of its dimension. In contrast to the general discussion about the dimension of different probe YDs in section 2.2 where equation (2.11) was mainly used, we will instead be using equation (2.12) in this appendix. We summarized the results at the end of this appendix for the convenience of the reader. This allows for an easier comparison with the general discussion in section 2.2.

Let us first start by introducing this family of YDs. The rows of a YD in this family are grouped into groups of  $n_0$  rows of equal length. Two consecutive groups will have a constant shift in length given by  $d_0$ . The number of sets of equal length rows will be  $m \leq [N/n_0]$  and the last rows will be of length  $d_0$ . The total number of boxes is:

$$h = \frac{1}{2} n_0 d_0 m (m + 1) . \quad (\text{D.1})$$

Let us denote by  $0 \leq k \leq (m - 1)$ , the group number and by  $1 \leq a \leq n_0$  the row number in each group, then the row length takes the form:

$$\psi_{k n_0 + a} = d_0 (m - k) ; \quad \text{if } 1 \leq k \leq m - 1 ,$$

and vanishes for  $k \geq m$ . To make connection with our conventions in appendix-A, notice that the number of rows of a YD in this family is  $n = n_0 m$ , and the number of columns is  $d = d_0 m$ . Notice also that the YDs in this family are homogeneous as can be checked easily even though not all of the columns/rows scale in the same way with  $N$ .

We will start our discussion by looking for an approximate expression of the log of the numerator of equation (2.12) for this family in the case where  $N$ ,  $n_0$ ,  $d_0$ , are very large. From now on we will call this numerator  $\text{num } \psi_{opt}$ . Before that, let us first fix the parametrization of the position of a box in a YD of this family. Since the YD is divided into  $m$  blocks of  $n_0$  rows and  $m$  blocks of  $d_0$  columns, one will need four integers  $i$ ,  $j$ ,  $s$ , and  $r$  where  $0 \leq i \leq m - 1$ ,  $0 \leq j \leq m - 1 - i$ ,  $1 \leq s \leq n_0$ , and  $1 \leq r \leq d_0$  to parametrize a position of a box. The latter will be characterized by its row number  $i n_0 + s$ , and its column number  $j d_0 + r$ . We will denote such information by  $(i, j; s, r)$ . Before continuing, let us also introduce a set of short notations:

$$\begin{aligned} \tilde{m} &= n_0 + d_0 , & \tilde{\psi} &= m \tilde{m} = d + n , & d &= d_0 m , & n &= n_0 m , \\ \tilde{N} &= N - n , & \overline{N} &= N + d = \tilde{N} + m \tilde{m} . \end{aligned} \quad (\text{D.2})$$

We will subdivide this family of YDs into two subfamilies corresponding to the case  $m \sim N^0$  and the case  $m \sim N^\mu$ ,  $\mu > 0$  and we will discuss them independently. We will call the first subfamily  $opt_0$  and the second subfamily  $opt_\mu$ . Before specializing to the two subfamilies, we will approximate  $\log \text{num } \psi_{opt}$  and  $\log \mathcal{H}_{opt}$  by carrying out all the sums involved except for the sum that depends on  $m$ .

As declared above, we start by evaluating  $\log \text{num } \psi_{opt}$ . Using the parametrization of

the position of boxes in  $\psi_{opt}$  we get:

$$\begin{aligned} \log \text{ num } \psi_{opt} &= \sum_{i=0}^{m-1} \sum_{j=0}^{m-1-i} \sum_{s=1}^{n_0} \sum_{r'=1}^{d_0} \log(N + j d_0 + r' - i m - s) \\ &= \sum_{k=0}^{m-1} \left[ \sum_{t=0}^{n_0-1} \sum_{u=0}^{(k+1)d_0-1} \log(\tilde{N} + k n_0 + t + u + 1) \right] \end{aligned}$$

where we introduced the new variables  $k = m - 1 - i$ ,  $t = n_0 - s$ ,  $r = r' - 1$ , and  $u = d_0 j + r$ . Notice that the sum inside the two brackets is of the form of the sum (B.9) whose approximate expression is given by (B.13). Using this observation, we get:

$$\begin{aligned} \log \text{ num } \psi_{opt} &\approx -\frac{3}{2}h + \frac{1}{2} \sum_{k=0}^{m-1} \left( \left[ (\tilde{N} + (k+1)\tilde{m})^2 - \frac{1}{6} \right] \log(\tilde{N} + (k+1)\tilde{m}) \right. \\ &\quad \left. - \left[ (\tilde{N} + (k+1)\tilde{m} - n_0)^2 - \frac{1}{6} \right] \log(\tilde{N} + (k+1)\tilde{m} - n_0) \right) \\ &\quad + \frac{1}{2} \left( \left[ \tilde{N}^2 - \frac{1}{6} \right] \log \tilde{N} - \left[ N^2 - \frac{1}{6} \right] \log N \right). \end{aligned} \quad (\text{D.3})$$

Next, we need to evaluate  $\log \mathcal{H}_{opt}$ . Once gain our starting point is the parametrization of the position of boxes in a YD of this family. It is easy to see that the hook length associated to a box  $(i, j; s, r)$  is given by:

$$h_{(i,j;s,r)} = [m - (i + j)] \tilde{m} + 1 - (r + s),$$

where  $1 \leq s \leq n_0$ ,  $1 \leq r \leq d_0$ , and  $i + j \leq m - 1$ . So we need to evaluate:

$$\begin{aligned} \log \mathcal{H}_{opt} &= \sum_{i=0}^{m-1} \sum_{j=0}^{m-1-i} \sum_{s=1}^{n_0} \sum_{r=1}^{d_0} \log([m - (i + j)] \tilde{m} + 1 - (r + s)) \\ &= \sum_{k=0}^{m-1} \sum_{u=0}^{n_0-1} \sum_{v=0}^{d_0-1} (n_0 - k) \log(k \tilde{m} + 1 + u + v) \\ &\approx -\frac{3}{2}h + \frac{1}{2} \sum_{k=0}^{m-1} (m - k) \left\{ \left[ (k+1)^2 \tilde{m}^2 - \frac{1}{6} \right] \log[(k+1)\tilde{m}] \right. \\ &\quad \left. - \left[ (k\tilde{m} + n_0)^2 - \frac{1}{6} \right] \log(k\tilde{m} + n_0) - \left[ (k\tilde{m} + d_0)^2 - \frac{1}{6} \right] \log(k\tilde{m} + d_0) \right\} \\ &\quad + \frac{1}{2} \sum_{k=1}^{m-1} (m - k) \left[ k^2 \tilde{m}^2 - \frac{1}{6} \right] \log(k\tilde{m}), \end{aligned} \quad (\text{D.4})$$

where we introduced the new variables  $k = m - 1 - (i + j)$ ,  $u = n_0 - s$ , and  $v = d_0 - r$  to move from the first to the second line. The evaluation of the sum over  $u$  and  $v$  follows the same steps as in evaluating  $\mathcal{H}_0$  defined in equation (B.9) whose approximate expression is given by equation (B.13).

We have two options in evaluating the leftover sum over  $k$  in the equations (D.3) and (D.4). We can either look for an approximate expression of the sum (B.1) for  $\ell \leq 3$ , perform

the sum over  $k$ , then expand the final result depending on the regime of the parameters  $n_0$ ,  $d_0$ , and  $m$ . Or, we can first expand the summands in equations (D.3) and (D.4), then perform the sum over  $k$  using, at worst, the approximations (B.6), (B.7) and (B.8). We find it easier to follow the second option and that is what we are going to do in the following for each subfamily. Another reason to follow such route is that we want to make our *opt* family of YDs as large as possible. In doing so the number  $m$  can be finite and small and hence the techniques of section B.1 of appendix-B will not be applicable. Before going ahead with our task, Let us introduce some notations for the sake of unifying the discussion below. We will assume the following leading behavior of  $n_0$ ,  $m$  and  $d_0$ :

$$\begin{aligned} n_0 &\approx \bar{n} N^\nu, \quad m \approx \bar{m} N^\mu, \quad d_0 \approx \bar{d} N^\delta, \\ n &\approx \bar{n} \bar{m} N^{\mu+\nu}, \quad d \approx \bar{m} \bar{d} N^{\mu+\delta}, \quad h \approx \bar{h} N^{2\mu+\nu+\delta}, \end{aligned} \quad (\text{D.5})$$

where  $\bar{n}$ ,  $\bar{m}$ ,  $\bar{d}$  are  $N$ -independent and  $\bar{h} = (1/2) \bar{n} \bar{d} \bar{m} (\bar{m} + 1)$ . Remember that we are interested in the regime of parameters:

$$2\mu + \nu + \delta < 2, \quad \nu + \mu \leq 1. \quad (\text{D.6})$$

### D.1 The $opt_0$ Family

This case corresponds to  $\mu = 0$ . In the present case our regime of parameters (D.6) simplifies to:

$$\nu + \delta < 2, \quad \nu \leq 1,$$

and since  $N$  enters in the expression (D.3), we distinguish the following cases:

- $1 < \delta < 2$ : This implies that  $\nu < 1$ , and hence  $\tilde{m} \approx d_0$ . We get:

$$\log \text{num } \psi_{opt} \approx \delta h \log N - h + \frac{2h}{m(m+1)} \left( \sum_{k=1}^m k \log(k \bar{d}) \right) + m \bar{n} N^{1+\nu} \log \frac{d_0}{N}, \quad (\text{D.7})$$

$$\log \mathcal{H}_{opt} \approx \delta h \log N - h + \frac{2h}{m(m+1)} \left( \sum_{k=1}^m k \log(k \bar{d}) \right) + \frac{1}{2} (\delta - \nu) N^{2\nu} \log N, \quad (\text{D.8})$$

$$\log \dim_N \psi_{opt} \approx (\delta - 1) m \bar{n} N^{1+\nu} \log N \approx n N \log \frac{d}{N}. \quad (\text{D.9})$$

- $\delta = 1$ : Once again we have  $\nu < 1$  and hence  $\tilde{m} \approx d_0$ . We find:

$$\log \text{num } \psi_{opt} \approx h \log N - h + \frac{2h}{\bar{d} m(m+1)} \left[ \sum_{k=1}^m (1 + k \bar{d}) \log(1 + k \bar{d}) \right], \quad (\text{D.10})$$

$$\log \mathcal{H}_{opt} \approx h \log N - h + \frac{2h}{m(m+1)} \left( \sum_{k=1}^m k \log(k \bar{d}) \right), \quad (\text{D.11})$$

$$\log \dim_N \psi_{opt} \approx \frac{2h}{\bar{d} m(m+1)} \sum_{k=1}^m [(1 + k \bar{d}) \log(1 + k \bar{d}) - (k \bar{d}) \log(k \bar{d})]. \quad (\text{D.12})$$

- $\delta < 1$ : We need to distinguish between four subcases:

\*  $\nu < \delta$ : This is the simplest case as  $\tilde{m} \approx d_0$  as before. We find:

$$\log \text{num } \psi_{opt} \approx h \log N + 0 h , \quad (\text{D.13})$$

$$\log \mathcal{H}_{opt} \approx \delta h \log N - h + \frac{2h}{m(m+1)} \left( \sum_{k=1}^m k \log(k \bar{d}) \right) , \quad (\text{D.14})$$

$$\log \dim_N \psi_{opt} \approx (1 - \delta) h \log N \approx h \log \frac{N}{d} , \quad (\text{D.15})$$

$$\begin{aligned} \log \dim_h \psi_{opt} &\approx \nu h \log N + \left( \log \left[ \frac{1}{2} \bar{n} m (m+1) \right] - \frac{2}{m(m+1)} \sum_{k=1}^m k \log k \right) h \\ &\approx h \log \frac{h}{d} + a h , \end{aligned} \quad (\text{D.16})$$

where  $a$  is some constant. The reason we included the next subleading term in the last expression is that  $\nu = 0$  is a valid point in our space of parameters. It is easy to check that this subleading term is positive when  $\nu = 0$  as it should be. This is because in this case  $\bar{n} \geq 1$  and:

$$\sum_{k=1}^m k \log k \leq \sum_{k=1}^m k \log m = \frac{1}{2} m (m+1) \log m .$$

\*  $\nu = \delta$ : This case is slightly more complicated than the previous one. We find:

$$\log \text{num } \psi_{opt} \approx h \log N + 0 h , \quad (\text{D.17})$$

$$\log \mathcal{H}_{opt} \approx \delta h \log N + \mathcal{O}(h) , \quad (\text{D.18})$$

$$\log \dim_N \psi_{opt} \approx (1 - \delta) h \log N \approx h \log \frac{N}{d} \approx h \log \frac{N}{n} , \quad (\text{D.19})$$

$$\log \dim_h \psi_{opt} \approx \delta h \log N \approx h \log \frac{h}{d} \approx h \log \frac{h}{n} , \quad (\text{D.20})$$

where we used that  $\nu = \delta \neq 0$ .

\*  $1 > \nu > \delta$ : In this case we have  $\tilde{m} \approx n_0$ . We find:

$$\log \text{num } \psi_{opt} \approx h \log N + 0 h , \quad (\text{D.21})$$

$$\log \mathcal{H}_{opt} \approx \nu h \log N - h + \frac{2h}{m(m+1)} \left( \sum_{k=1}^m k \log(k \bar{n}) \right) , \quad (\text{D.22})$$

$$\log \dim_N \psi_{opt} \approx (1 - \nu) h \log N \approx h \log \frac{N}{n} , \quad (\text{D.23})$$

$$\begin{aligned} \log \dim_h \psi_{opt} &\approx \delta h \log N + \left( \log \left[ \frac{1}{2} \bar{d} m (m+1) \right] - \frac{2}{m(m+1)} \sum_{k=1}^m k \log k \right) h \\ &\approx h \log \frac{h}{n} + a h , \end{aligned} \quad (\text{D.24})$$

where  $a$  is a constant. Notice that the roles of  $n_0$  and  $d_0$  here are switched with respect to the case  $\delta > \nu$ , even though the approximate expression (D.3) is not



symmetric under the exchange  $n_0 \leftrightarrow d_0$ . This is a manifestation the duality discussed in section-2.2.1.

\*  $1 = \nu > \delta$ : In this case  $\tilde{m} \approx n_0 \sim N$ . We find:

$$\log \text{num } \psi_{opt} \approx h \log N - h - \frac{2h}{\bar{m} m (m+1)} \left[ \sum_{k=1}^m (1 - k \bar{n}) \log(1 - k \bar{n}) \right] , \quad (\text{D.25})$$

$$\log \mathcal{H}_{opt} \approx h \log N - h + \frac{2h}{m(m+1)} \left( \sum_{k=1}^m k \log(k \bar{n}) \right) , \quad (\text{D.26})$$

$$\log \dim_N \psi_{opt} \approx -\frac{2h}{\bar{n} m (m+1)} \sum_{k=1}^m [(1 - k \bar{n}) \log(1 - k \bar{n}) + (k \bar{n}) \log(k \bar{n})] . \quad (\text{D.27})$$

Notice that  $\log \dim_N \psi_{opt}$  is well defined and positive since  $m \bar{n} < 1$

## D.2 The $opt_\mu$ Family

In this case, we have the following range of parameters:

$$0 < \mu \leq 1 , \quad 0 < \nu + \mu \leq 1 , \quad 0 < 2\mu + \nu + \delta < 2 .$$

The origin of the complication in this case is that we need to worry about  $d = m d_0$  and  $n = n_0 m$  on top of  $n_0$  and  $d_0$ . As in the previous subfamily we will use the different ranges of  $\delta$  as the main classifying tool. We have the following cases:

- $1 < \delta < 2$ : This is the easiest case. We find:

$$\begin{aligned} \log \text{num } \psi_{opt} \approx & \left( h + n N - \frac{1}{2} n^2 \right) \log \tilde{\psi} + \frac{h}{6 m^2} \log m \\ & - n \left( N - \frac{1}{2} n \right) \log N - \frac{3}{2} h + \frac{h}{2 m} + \dots , \end{aligned} \quad (\text{D.28})$$

$$\log \mathcal{H}_{opt} \approx \left( h - \frac{1}{2} n_0 n \right) \log \tilde{\psi} + \frac{h}{6 m^2} \log m - \frac{1}{2} n_0 n \log n_0 - \frac{3}{2} h + \frac{h}{2 m} + \dots , \quad (\text{D.29})$$

$$\log \dim_N \psi_{opt} \approx (\mu + \delta - 1) \bar{n} \bar{m} N^{1+\nu+\mu} \log N \approx n N \log \frac{d}{N} . \quad (\text{D.30})$$

- $\delta = 1$ : Following the same steps as before we find:

$$\begin{aligned} \log \text{num } \psi_{opt} &\approx \left( h + \frac{h}{6m^2} + nN - \frac{1}{2}n^2 \right) \log \tilde{\psi} \\ &\quad - \left( \frac{1}{6} + \frac{1+\bar{d}}{(\bar{d})^2} \right) \left( \frac{h}{m^2} \right) \log(N + d_0) \\ &\quad - n \left( N - \frac{1}{2}n \right) \log N - \frac{3}{2}h + \frac{h}{2m} + \dots, \end{aligned} \quad (\text{D.31})$$

$$\log \mathcal{H}_{opt} \approx \left( h - \frac{1}{2}n_0 n \right) \log \tilde{\psi} + \frac{h}{6m^2} \log m - \frac{1}{2}n_0 n \log n_0 - \frac{3}{2}h + \frac{h}{2m} + \dots, \quad (\text{D.32})$$

$$\log \dim_N \psi_{opt} \approx \mu \bar{m} \bar{n} N^{1+\mu+\nu} \log N \approx nN \log \frac{d}{N}. \quad (\text{D.33})$$

- $\delta < 1$ : As before we distinguish three subcases:

★  $\nu < \delta$ : We distinguish three possibilities:

\*  $\delta + \mu > 1$ : In this case we need to keep terms that involve  $d_0$  in the log expression. We find:

$$\log \text{num } \psi_{opt} \approx (h + nN) \log \tilde{\psi} - nN \log N - \frac{3}{2}h + \dots, \quad (\text{D.34})$$

$$\log \mathcal{H}_{opt} \approx \left( h - \frac{1}{2}n_0 n \right) \log \tilde{\psi} - \frac{3}{2}h + \frac{h}{2m} + \dots, \quad (\text{D.35})$$

$$\log \dim_N \psi_{opt} \approx (\delta + \mu - 1) \bar{m} \bar{n} N^{1+\mu+\nu} \log N \approx nN \log \frac{d}{N}. \quad (\text{D.36})$$

\*  $\delta + \mu = 1$ : Following the same steps as in the previous case, we find:

$$\begin{aligned} \log \text{num } \psi_{opt} &\approx \left( h - \frac{1}{2}n^2 \right) \log N + \left( \frac{1+\bar{\psi}}{\bar{\psi}} \right)^2 h \log(1 + \bar{\psi}) \\ &\quad - \left( \frac{3}{2} + \frac{1}{(\bar{\psi})} \right) h + \dots, \end{aligned} \quad (\text{D.37})$$

$$\log \mathcal{H}_{opt} \approx \left( h - \frac{1}{2}n_0 n \right) \log \tilde{\psi} - \frac{3}{2}h + \dots, \quad (\text{D.38})$$

$$\log \dim_N \psi_{opt} \approx \frac{1}{(\bar{\psi})^2} [(1 + \bar{\psi})^2 \log(1 + \bar{\psi}) - (\bar{\psi})^2 \log \bar{\psi} - \bar{\psi}] h, \quad (\text{D.39})$$

where  $\bar{\psi} = \bar{m} \bar{d}$ .

\*  $\delta + \mu < 1$ : We find in this case:

$$\log \text{num } \psi_{opt} \approx h \log N + 0h + \dots, \quad (\text{D.40})$$

$$\log \mathcal{H}_{opt} \approx \left( h - \frac{1}{2}n_0 n \right) \log \tilde{\psi} - \frac{3}{2}h + \dots, \quad (\text{D.41})$$

$$\log \dim_N \psi_{opt} \approx (1 - \mu - \delta) h \log N \approx h \log \frac{N}{d}, \quad (\text{D.42})$$

$$\log \dim_h \psi_{opt} \approx (\nu + \mu) h \log N \approx h \log \frac{h}{d}. \quad (\text{D.43})$$

★  $\delta = \nu$ : We find:

$$\log \text{num } \psi_{opt} \approx h \log N - \frac{3}{2} \left( \frac{\bar{n}}{d} \right) h + \dots, \quad (\text{D.44})$$

$$\log \mathcal{H}_{opt} \approx h \log \tilde{\psi} + \dots, \quad (\text{D.45})$$

$$\log \dim_N \psi_{opt} \approx (1 - \mu - \delta) h \log N \approx h \log \frac{N}{d} \approx h \log \frac{N}{n}, \quad (\text{D.46})$$

$$\log \dim_h \psi_{opt} \approx (\mu + \delta) h \log N \approx h \log \frac{h}{d} \approx h \log \frac{N}{n}. \quad (\text{D.47})$$

★  $\delta < \nu$ : We distinguish two possibilities:

\*  $\mu + \nu = 1$ : In this case we need to keep terms involving  $n_0$  untouched when expanding the log terms. Then, we evaluate the sum over  $k$ . We find:

$$\log \text{num } \psi_{opt} \approx h \log N + \left( \frac{1 - \bar{m} \bar{n}}{\bar{m} \bar{n}} \right)^2 h \log(1 - \bar{m} \bar{n}) + \left( \frac{1}{\bar{m} \bar{n}} - \frac{3}{2} \right) h, \quad (\text{D.48})$$

$$\log \mathcal{H}_{opt} \approx h \log \tilde{\psi} - \frac{3}{2} h + \dots, \quad (\text{D.49})$$

$$\log \dim_N \psi_{opt} \approx \frac{h}{(\bar{m} \bar{n})^2} \left[ (1 - \bar{m} \bar{n})^2 \log(1 - \bar{m} \bar{n}) - (\bar{m} \bar{n})^2 \log(\bar{m} \bar{n}) + (\bar{m} \bar{n}) \right]. \quad (\text{D.50})$$

Notice that  $\dim_N \psi_{opt}$  is positive as it should be. To see that remember that<sup>16</sup>  $(\bar{m} \bar{n}) \leq 1$ .

\*  $\mu + \nu < 1$ : Here we expand the log terms keeping only  $N$ . We find, after summing over  $k$ :

$$\log \text{num } \psi_{opt} \approx h \log N + \dots, \quad (\text{D.51})$$

$$\log \mathcal{H}_{opt} \approx h \log \tilde{\psi} - \frac{3}{2} h + \dots, \quad (\text{D.52})$$

$$\log \dim_N \psi_{opt} \approx (1 - \nu - \mu) h \log N \approx h \log \frac{N}{n}, \quad (\text{D.53})$$

$$\log \dim_h \psi_{opt} \approx (\mu + \delta) h \log N \approx h \log \frac{h}{n}. \quad (\text{D.54})$$

### D.3 Summary

In the following we summarize what we got in the previous two subsections. This will make it much easier to compare with the general discussion in section 2.2. To make contact with the expressions there we introduce the quantity:

$$\psi_0 = \text{Max} \{d, n\}.$$

We will also borrow the classification of probes from there. Looking at the different cases above, we find the following leading behavior of  $\log \dim_N \psi$  and  $\log \dim_h \psi$ :

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<sup>16</sup>It is easy to prove that the function:  $f(x) = (1 - x)^2 \log(1 - x) - x \log x + x$ , is an increasing function in the interval  $(0, 1]$  which implies that  $0 \leq f(x) \leq 1$ .

- Generic Probes: In the case  $\mu = 0$ , these probes correspond to  $\delta < 1$  and include the cases  $\delta < \nu$ ,  $\delta = \nu$  and  $\delta > \nu$ . In the case  $\mu \neq 0$ , these probes correspond to  $\delta < 1$  once again and include the cases  $(\nu < \delta, \mu + \delta < 1)$ ,  $\nu = \delta$  and  $(\delta < \nu, \nu + \mu < 1)$ . In all these cases, we find that:

$$\log \dim_N \psi \approx h \log \frac{N}{\psi_0} \quad , \quad \log \dim_h \psi \approx h \log \frac{h}{\psi_0} + a h \quad , \quad (\text{D.55})$$

where  $a$  is some constant whose precise value is not of interest to us.

- Linear Probes: In the case  $\mu = 0$ , these probes include  $\delta = 1$  and  $\nu = 1$  cases. In the case  $\mu \neq 0$ , we have the cases  $(\nu < \delta, \mu + \delta = 1)$  and  $(\delta < \nu, \nu + \mu = 1)$ . In all this cases we find:

$$\log \dim_N \psi \approx b h \quad , \quad (\text{D.56})$$

where  $b$  is some constant whose value is not important to us.

- Long Probes: In the case  $\mu = 0$ , these probes correspond to  $\delta > 1$ . In the other case  $\mu \neq 0$ , the probes correspond to the cases  $\delta > 1$ ,  $\delta = 1$ , and  $(\delta < 1, \delta + \mu > 1)$ . In all of these cases, we find:

$$\log \dim_N \psi \approx n N \log \frac{d}{N} \quad . \quad (\text{D.57})$$

These results are in complete agreement with what we found in the general discussion in section-2.2.

## E. Some useful properties of Kostka numbers

In this appendix we will discuss some of the properties of Kostka numbers  $K_{\psi, \beta}$  that will be very useful in subsections 4.2 and 4.3. In the following the filling  $\beta$  has no zero entries, see the end of subsection 4.1.2 for more details.

### E.1 Kostka numbers and fillings

The first property we are going to discuss has to do with the behavior of the Kostka numbers under the reshuffling of the numbers  $\beta_i$ 's that define a filling  $\beta$ . The claim is that the Kostka numbers are invariant under that and hence we can take  $\tilde{\beta}$  the ordered counterpart of  $\beta$  as a representative of these fillings i.e. start from a filling  $\beta = (\beta_1, \beta_2, \dots, \beta_m)$ , and construct the filling  $\tilde{\beta} = (\tilde{\beta}_1, \tilde{\beta}_2, \dots, \tilde{\beta}_m)$  such that:

$$\tilde{\beta}_1 \geq \tilde{\beta}_2 \geq \dots \geq \tilde{\beta}_m \quad .$$

To prove the claim above it is enough to prove that we can construct a one-to-one map between the SSYT $\alpha$   $\psi$  with fillings  $\beta$  and  $\tilde{\beta}$  that are related by exchanging two numbers  $\beta_i$  and  $\beta_{i+1}$ . i.e.

$$\beta = (\beta_1, \beta_2, \dots, \beta_i, \beta_{i+1}, \dots, \beta_m) \quad , \quad \tilde{\beta} = (\beta_1, \beta_2, \dots, \beta_{i+1}, \beta_i, \dots, \beta_m) \quad .$$

Let us assume that we constructed all the SSYT $\alpha$   $\alpha$  with filling  $\beta$  and we want to construct the ones associated to the filling  $\tilde{\beta}$ . This is done in three steps:

1. First, we relabel by  $(i + 1)$  the boxes labeled by  $i$ , and vice versa. Notice that since we are not touching the other labels in this step, we need only to deal with the order of the labels  $i$  and  $(i + 1)$ .
2. Next, we first deal with the order of the labels  $i$  and  $(i + 1)$  in each column. If the label  $i$  sits below the label  $(i + 1)$  we exchange their position, otherwise we leave the column unchanged. Notice that in this step if the column is not left unchanged then all we have done is switching the labels of only two boxes in this column.
3. Finally, we look at the order of labels in each row. Once again we need to reorder the boxes with labels  $i$  and  $(i + 1)$  only. This is done by putting the boxes with label  $i$  that are to the right of the boxes with labels  $(i + 1)$  to their left. At the end of this step we get a SSYT  $\alpha$  with filling  $\bar{\beta}$  for each SSYT  $\alpha$  with filling  $\beta$ .

Notice that we could have done the same by starting with the filling  $\bar{\beta}$ , then construct the semi-standard Young tableaux associated to the filling  $\bar{\beta}$ . Hence the two Kostka numbers associated to  $\beta$  and  $\bar{\beta}$  are equal.

Take for example the fillings  $\beta = (1, 1, 2, 1)$  and  $\bar{\beta} = (1, 2, 1, 1)$  and the YD:

$$\begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline \end{array}.$$

The only SSYTx associated to the filling  $\beta$  are:

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 3 & 4 & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|} \hline 1 & 3 & 3 \\ \hline 2 & 4 & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 3 & \\ \hline \end{array}.$$

Let us follow the steps above to construct the SSYTx associated to  $\bar{\beta}$ . The first step gives:

$$\begin{array}{|c|c|c|} \hline 1 & 3 & 2 \\ \hline 2 & 4 & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|} \hline 1 & 2 & 2 \\ \hline 3 & 4 & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 2 & \\ \hline \end{array}.$$

The only YT that is affected by the second step is the last one, so we have:

$$\begin{array}{|c|c|c|} \hline 1 & 3 & 2 \\ \hline 2 & 4 & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|} \hline 1 & 2 & 2 \\ \hline 3 & 4 & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 2 & 3 & \\ \hline \end{array}.$$

Finally, the last step remedies the first YT which is the only non-SSYT. We get at the end the following SSYTx associated to the filling  $\bar{\beta} = (1, 2, 1, 1)$ :

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 2 & 4 & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|} \hline 1 & 2 & 2 \\ \hline 3 & 4 & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 2 & 3 & \\ \hline \end{array}.$$

It is not hard to see that the SSYT above are the only ones possible for the filling  $\bar{\beta}$ . In the remaining of this appendix whenever we talk about a filling  $\beta$ , we assume that it is completely ordered i.e.  $\beta_1 \geq \beta_2 \geq \dots$ .

## E.2 The non-zero Kostka numbers

The question we want to address here is when is the Kostka number non vanishing? Since Kostka numbers  $K_{\psi, \beta}$  are related to SSYT $\times \psi$  which are themselves related to decomposition of the tensor product  $\psi \otimes B$ , where  $B$  is some arbitrary YD<sup>17</sup>, through the ordering rule (see section 4.1.1) according to our discussion in section 4.1.2, we can use the implications of the ordering rule (see the end of section 4.1.1) as a guiding tool.

The first implication of the ordering rule is that a box in row  $i$  of the YD  $\psi$  cannot be attached to a row  $j$  of the YD  $B$  if  $i < j$ , see end of section 4.1.1. This can be easily understood in the case of SSYT and has to do with the fact that the labels in the same column should be strictly increasing from top to bottom and labels in the same row should be weakly increasing from left to right. In the language of SSYT,  $i$  will be the row number and  $j$  will be the label. Due to the aforementioned conditions on the labels in a SSYT, the labels in row  $i$  should be bigger or equal than  $i$ , which implies the absence of the label  $j$  in row  $i$  if  $j < i$ .

The second implication of the ordering rule (see end of section 4.1.1) can also be understood as a consequence of the conditions of the labels of SSYT $\times$ . In terms of the tensor product decomposition, it was shown that if the boxes added to a row  $i$  of  $B$  come from different rows of  $\psi$  with  $m$  being the highest one among them, then the number of added boxes to the row  $i$  is less or equal than  $\psi_m$  the length of this row. In terms of the SSYT labeling, this situation corresponds to the presence of the label  $i$  in the row  $m$  of  $\psi$ . Due to the conditions on the labeling of SSYT, any other box with label  $i$  should be to the left of this box even if it belongs to a row below row  $m$ . As a result the number of labels  $i$  in the SSYT  $\psi$  is less or equal  $\psi_m$  the length of the row  $m$  of  $\psi$ .

The combination of the previous two conditions, together with the fact that the filling  $\beta$  is a totally ordered filling, implies the following condition for a non-vanishing Kostka number  $K_{\psi, \beta}$ , see for example [33]. The Kostka number  $K_{\psi, \beta}$  is nonzero if and only if both  $\psi$  and  $\beta$  are partitions of the same integer  $h$  and moreover  $\psi$  is larger than  $\beta$  in the dominance order. The latter is defined as follows. Let  $\psi = (\psi_1, \psi_2, \dots, \psi_n)$ ,  $\beta = (\beta_1, \beta_2, \dots, \beta_m)$  with  $\psi_1 \geq \psi_2 \geq \dots$  and  $\beta_1 \geq \beta_2 \geq \dots$  be two ordered tuples of integers. We say that  $\psi$  is larger than  $\beta$  in dominance order, and we write  $\psi \succeq \beta$ , if and only if for each integer  $k \geq 1$ , the following is true:

$$\forall k \geq 1 ; \quad \sum_{i=1}^k \psi_i \geq \sum_{i=1}^k \beta_i ,$$

where we fill in the non-existing integers  $\psi_k$  for  $k > n$  by zeros. An immediate consequence of this is that the number of entries in  $\beta$  should be bigger or equal to the number of rows of  $\psi$ . Notice that this condition is only true for  $\beta$  being totally ordered. If  $\beta$  is not ordered then the statement above works in one direction only:  $K_{\psi, \beta}$  nonzero then we have the dominance order condition but not the way around.

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<sup>17</sup>We choose  $B$  such that the number of its columns is large enough.

### E.3 Maximizing Kostka numbers

The final point in our investigation on the Kostka numbers is to find conditions on the filling  $\beta$  such that we get a maximum value for  $K_{\alpha, \beta}$ . This is important as we are interested in leading order behavior of the degeneracy of YD appearing in the decomposition of  $\mathcal{O} \otimes \psi$ , and as we will see in subsection 4.3 and section F.2 of appendix-F, this degeneracy is intimately connected to the maximum Kostka number in some cases of interest to us. Notice that we are fixing the shape  $\psi$  here and varying  $\beta$ . Two claims can be made here. First of all,  $|\beta|$  the “length” of the filling  $\beta$  should be maximized. By length we mean the number of non-zero entries  $\beta_i \neq 0$ . Secondly, the entries  $\beta_i$  should be very close to the average  $h/|\beta_0|$ <sup>18</sup>. In the following  $h$  will stand, as usual, for the number of boxes of the tableau  $\psi$  whose shape is also denoted by  $\psi$ , whereas  $N$  is, as usual, the flux of the background geometry which is the same  $N$  that appears in the group of our interest  $SU(N)$ .

The first claim is easy to understand. The condition on the columns ordering implies that we cannot have the same label on more than one box in the same column. Hence, to have more options in labeling we need to have more labels. As a result the bigger  $|\beta|$  the more options we have. In the case  $N \leq h$ , the biggest  $|\beta|$  possible is  $h$  which implies that  $\beta_i = 1$ . So, the condition on  $|\beta|$  is strong enough to imply the second claim above. Things are more complicated in the case  $N < h$  since here we have at least one  $\beta_i$  that is bigger than “1” for fillings associated with the largest possible value of  $|\beta| = N$ . In this case the second claim adds a non trivial condition on the filling  $\beta$ . The argument for the validity of the second claim in this case is as follows.

First of all we are going to work with the ordered filling  $\tilde{\beta}$ . We will also concentrate on the case  $N < h$ . Hence we are starting with the  $N$ -tuple  $\tilde{\beta} = (\tilde{\beta}_1, \tilde{\beta}_2, \dots, \tilde{\beta}_N)$  such that:

$$\tilde{\beta}_1 \geq \tilde{\beta}_2 \geq \dots \geq \tilde{\beta}_N, \quad \sum_{i=1}^N \tilde{\beta}_i = h.$$

Let us first concentrate on  $\tilde{\beta}_1$  and  $\tilde{\beta}_2$ . Let us fix the sum  $\tilde{\beta}_1 + \tilde{\beta}_2$  but allow each of them to vary. Notice that the boxes with label 1 are confined to the first row whereas the ones with label 2 have the freedom to be located in either of the first two rows. So, the bigger  $\tilde{\beta}_2$  the bigger the Kostka number. We reach the maximum if  $\tilde{\beta}_2 \approx \tilde{\beta}_1$  since  $\tilde{\beta}_2 \leq \tilde{\beta}_1$ . Repeating the same argument for the labels  $i$  for  $3 \leq i \leq n$  where we use that boxes with label  $j$  are confined to be in the first  $j$  rows, we conclude that to maximize the Kostka number we should choose:

$$\tilde{\beta}_1 \approx \tilde{\beta}_2 \approx \dots \approx \tilde{\beta}_n.$$

To complete the argument we start from below. Once again fixing the sum  $\tilde{\beta}_N + \tilde{\beta}_{N-1}$ , but allowing both of them to vary. The position of the boxes with label  $N$  is at the end of the rows with the condition that there are no boxes below them. For the boxes with label

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<sup>18</sup>An intuitive argument to why such a choice is special is as follows. We argued before that the Kostka numbers are invariant under permutations of the entries of the filling  $\beta$ . The fixed point of such a permutation is when all the entries are equal which reproduces precisely this choice.

$(N - 1)$ , they can be either on top of the boxes with label  $N$ , or to their left under the condition that if there are boxes below, they should carry the label  $N$ . So, if we have in a row  $k$  boxes with label  $N$ , we have at least  $k$  possible boxes to carry the label  $(N - 1)$ . Hence the bigger  $\tilde{\beta}_N$ , the more options we have which leads to the optimum situation  $\tilde{\beta}_{N-1} \approx \tilde{\beta}_N$ . We can repeat the same argument for the other labels and reach a similar conclusion. Hence the second claim.

In the argument above, we cheated a little bit. Remember that there should be no boxes below the ones with label  $N$ . So the number of such boxes i.e. boxes with no box below in the YD, constitute an upper bound on  $\tilde{\beta}_N$ . So, the claim above should be modified accordingly. However, as we will see in subsection 4.3, our case of interest corresponds to  $h/N$  is small enough, together with the type of YD we are dealing with put us in a safe position.

## F. From Kostka numbers to actual degeneracies $d_k$

In the following we will discuss the possible modifications to the conclusions derived in subsections 4.2 and 4.3 for SSYT $\psi$  and their Kostka numbers once the YD and the antisymmetry rules are included to make contact with the tensor decomposition (3.3). Following the arguments for the Kostka numbers, the discussion depends on whether  $h \ll N$ , or  $(h \sim N, h \gg N)$ .

### F.1 The case $h \ll N$

We arrived in subsection 4.2 at the conclusion that, without the inclusion of the YD and the antisymmetry rule, we can take the YDs  $\varphi_k^0$  as representatives of the the YDs  $\varphi_k$  that appear in the decomposition of the tensor product  $\mathcal{O} \otimes \psi$ . Remember that the YDs  $\varphi_k^0$  are the result of adding at most one box to each row of  $\mathcal{O}$ . Their degeneracy  $d_k^0$  as well as their number  $\mathcal{N}_0$  have the leading terms given in equation (4.8), which we rewrite here for convenience.

$$\log d_k \approx h \log \frac{h}{\psi_0}, \quad \log \mathcal{N}_0 \approx h \log \frac{N}{h}. \quad (\text{F.1})$$

What happens when we take into account the YD and the antisymmetry rules? For concreteness, let us restrict ourselves to the special case of the background YD  $\mathcal{O}_0$  whose rows' length is given by:

$$\mathcal{O}_i = N - i; \quad i = 1, 2, \dots, N - 1.$$

This corresponds to the limit shape YD of the superstar ensemble where the number of columns of its YDs is fixed to be  $N$  as well. We will discuss the possible modifications that we need to take into account for the general case at the end.

Notice that for this background  $\mathcal{O}_0$  neither the YD rule nor the antisymmetry rule modify the results above since both are trivially satisfied. Hence there are  $\mathcal{N}_0$  YDs that are constructed by adding at most one box to each row, each of them has the degeneracy  $d_k^0$ , which in total reproduces the leading behavior<sup>19</sup> of  $\dim_N \psi$ . We will not discuss the

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<sup>19</sup>This is after taking the log of the numbers under consideration.



other YDs  $\varphi_k$  since they will not play a role in what we are trying to do, see subsection 5.1.1 for more details.

To close the line of thoughts, we need to discuss what happens in the case of a different background  $\mathcal{O}$ . We need to distinguish between a YD  $\mathcal{O}$  in the same ensemble as  $\mathcal{O}_0$ , and YD in a different ensemble. The worst scenario in both cases amounts to replacing  $N$  by  $\kappa N$ , with  $\kappa \leq 1$ . The idea is to replace every set of consecutive equal length rows with just one row, which brings our discussion close enough<sup>20</sup> to the one for  $\mathcal{O}_0$ . The only thing we need to make sure about is that after doing so, we are still left with order  $N$  rows. First, let us deal with a YD  $\mathcal{O}$  that belongs to the same ensemble as  $\mathcal{O}_0$ . In this case we know that the fluctuation in the row length is  $\delta \mathcal{O}_i \sim \sqrt{N}$  [1] (see also subsection 2.1). Hence, the number of consecutive rows of equal length can be at most of order  $\sqrt{N}$ . So, in general we can have  $N^a$ ,  $a \leq 1/2$  consecutive rows of equal length. For these set of rows, the number of moved boxes is of order  $N^{2a}/2$ . Since the total number of moved boxes in  $\mathcal{O}$  when compared with  $\mathcal{O}_0$  is of order  $N$  [1] (see also subsection 2.1), the number of such sets should grow slower than  $N^{1-a}$ . Hence once we collapse these sets to just one row we will end up with a YD whose number of rows is of order  $N$ . Applying the same considerations as in the case of  $\mathcal{O}_0$ , we conclude that the leading behavior of both the number of different YD in the tensor product decomposition as well as their degeneracy does not change, since the only thing one needs to do is to replace  $N$  there by  $\kappa N$  inside the log.

What about a YD from another ensemble? We start with its limit shape YD. We can have cases where this limit shape YD has sets of order  $N^0$  of equal length, for example  $N = 2D$ , where  $N$  is the number of rows whereas  $D$  is the number of columns. In these cases we should collapse these sets to just one row. In the opposite cases, we leave the limit shape YD untouched. All in all we end up with a YD where the number of its rows is of order  $N$ , and where the shift of length between two consecutive rows is at worst one box. As a result, we can borrow the discussion above to conclude that the leading behavior of the number of the different YDs in the tensor product decomposition as well as the leading behavior of their degeneracy remains intact. For the other YDs in this ensemble, we can easily adapt the discussion of the YD in the  $\mathcal{O}_0$  ensemble to this case, reaching the same conclusion.

All in all, in the tensor product decomposition of  $\mathcal{O} \otimes \psi$ , where  $\psi$  is a YD with number of boxes  $h \ll N$ , the number of different YDs and their degeneracy has the following leading behavior:

$$\log d_k \approx h \log \frac{h}{\psi_0}, \quad \log \mathcal{N} \approx h \log \frac{N}{h}, \quad (\text{F.2})$$

where, according to our conventions (appendix-A),  $\psi_0 = \text{Max}\{n, d\}$ ,  $d$  is the number of columns of  $\psi$ , and  $n$  is the number of its rows.

## F.2 The cases $h \sim N$ and $h \gg N$

We arrived in subsection 4.3 to the conclusion that, in the case where  $\psi$  is a generic probe and  $h \sim N$  or  $h \gg N$ , and forgetting about the antisymmetry rule, there are special YDs

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<sup>20</sup>In general, the shift in the length of consecutive rows can be more than one box which does not change the essence of the discussion in the case of  $\mathcal{O}_0$ .

$\varphi_k^*$  in the tensor product decomposition (3.3) whose degeneracy  $d_k^*$  has a leading behavior given by equation (4.10), which we rewrite here for convenience:

$$\log d_k^* \approx \dim_N \psi \approx h \log \frac{N}{\psi_0} . \quad (\text{F.3})$$

These YDs are constructed by adding  $h_i \approx (h/N)$  boxes to each row of the YD  $\mathcal{O}$ . Notice that  $h_i \ll d$  and  $h_i \ll n$  since  $n, d \ll N$  and  $h \sim n d$ . In this appendix, we want to study the implication of taking into account the antisymmetry rule. To fix notations below we will denote by  $\beta_0$  the filling of the SSYT  $\psi^*$  associated to one of the YDs  $\varphi_k^*$ . Hence:

$$\beta_0 = (h_1, h_2, \dots, h_N), \quad h_i \approx \frac{h}{N}, \quad h_1 \geq h_2 \geq \dots \geq h_N,$$

where  $h_i$  is as usual the degeneracy of the label  $i$ . We have the following leading behavior of the associated Kostka number:

$$\log K_{\alpha, \beta} \approx \dim_N \psi ,$$

We further concentrate on the simple case where the background YD  $\mathcal{O}_0$  whose rows' length is given by:

$$\mathcal{O}_i = N - i ; \quad i = 1, 2, \dots, N - 1 .$$

This the same background YD that we used in the explicit discussion in the previous section. We will come back to the possible modifications to the arguments below at the end of this section.

What does the antisymmetry rule imply? First of all, the labels associated to the first row should be different for each box except for the first few ones that carry the label 1. So, for  $i > 1$  we cannot find more than one box in the first row with this label. The same argument leads to the conclusion that in row  $k$ , there should be at most one box with label  $i > k$ . So essentially, taking into account the antisymmetry rule forces the labeling of the YD  $\psi^*$  to be of the following form. The label  $1 \leq i \leq n$  is mostly confined to the row  $i$ . In other words there will be at most one box with label  $j$  in a row  $i \neq j$ . The remaining labels  $i > n$  will appear at most once in each row. To proceed further, we write schematically the different contributions to the associated Kostka number as:

$$K_{\alpha, \beta_0} = K_{\alpha, \beta_0}^{(0)} + \sum_a K_{\alpha, \beta_0}^{(a)} , \quad (\text{F.4})$$

where  $K_{\alpha, \beta_0}^{(0)}$  stands for the contribution giving rise to the degeneracy we are after i.e. the labelings that satisfies the required conditions discussed above, and the sum is over the remaining “bad” contributions. We will call the former the good labelings and the latter the bad labelings. The meaning of the index  $a$  in the sum will be clear below, but at the moment it stands for some indexing of the bad labelings. Our aim in the following is to relate somehow the bad labelings to the good ones. But before that, let us first deal with the special labels  $\{1, 2, \dots, n\}$ . Remember that these labels are special in the sense that they are the only ones that are allowed to occur more than once given that they are in the

row with the same number as the label. In other words, we can have more than one box with label  $i$  if and only if  $1 \leq i \leq n$  and the associated boxes belong to row  $i$  in the tableau  $\psi$ . Let us count the number of labelings of  $\psi$  that violate this requirement. It is easy to see that this number is smaller than:

$$\log n_{\text{wrong}} = \sum_{i=1}^n \log C_{h_i+i-1}^{h_i} \sim \left(\frac{n}{N}\right) h \log N \ll h \log N .$$

Given that  $\log K_{\alpha, \beta_0} \sim h \log N$ , we can safely restrict ourselves to counting the number of labelings where the first  $n$  labels are fixed as above. Said differently, we will try in the following to estimate the number of labeling  $K_{\alpha, \beta_0}$  such that the labels  $1 \leq i \leq n$  are mainly confined to be each in its corresponding row  $i$ . Our discussion below will concern only the labels  $n < i \leq N$ .

A useful way to think about counting the labelings  $K_{\alpha, \beta_0}$  is as follows. First construct all the labelings where each row of  $\psi$  has at most one box with label  $i$ , for all the labels  $n < i \leq N$ . These labelings were called good labelings and their number  $K_{\alpha, \beta_0}^{(0)}$  is the one we are after. For each labelings among these, we will move the boxes around in order to increase the degeneracy of labels in the rows of  $\psi$ . For each good labeling  $a$ , the number of bad labelings constructed so  $\tilde{K}_{\alpha, \beta_0}^{(a)}$  is smaller than:

$$\mathcal{N}_a = \prod_{k=1}^N C_{2h_k-1}^{h_k} \approx 2^{2h} ,$$

where we used that  $\sum_k h_k = h$ . It is easy to convince yourself that in this way, we will be able to construct all the labeling of  $\psi$  associated to  $\beta_0$ . The reason is once again the inequality  $h_i \ll n$ . This is because, starting from a bad labeling, one can find a lot of possible rows to accommodate the degenerate labels in order to construct at the end a good labeling of  $\psi$ .

Now identifying the label  $a$  in the sum (F.4) with the index of good labels, together with the identification  $\tilde{K}_{\alpha, \beta_0}^{(a)} = K_{\alpha, \beta_0}^{(a)}$  and its upper bound above, we conclude that to leading order:

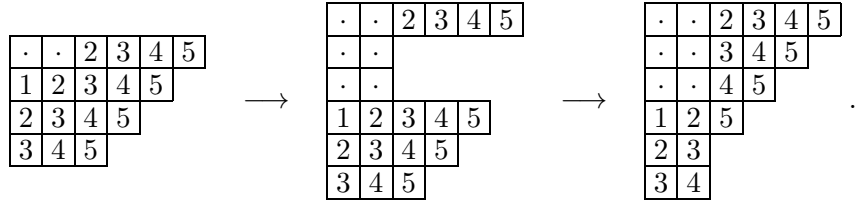
$$\log K_{\alpha, \beta_0}^{(0)} \approx \log K_{\alpha, \beta_0} \approx \log \dim_N \psi \sim h \log N . \quad (\text{F.5})$$

This result looks universal despite the fact that we started with a specific background tableau  $\mathcal{O}_0$ . Actually one can easily generalize the arguments above to other backgrounds, either belonging to the same ensemble as  $\mathcal{O}_0$ , or other ensemble. The key idea is the observation mentioned above regarding the scale of  $h_i$ . Remember we have  $h_i \ll n, d$ . Let us discuss each case on its own.

For other backgrounds in the ensemble of  $\mathcal{O}_0$ , the complication arises because in this case we do not have the simple picture of a jump by one box between consecutive rows as in  $\mathcal{O}_0$ . We could either have some rows with the same length, or the difference in length of some consecutive rows will be larger than one box. The first situation in principle reduces the number of possible YDs in the tensor decomposition, however we can easily see that the leading order does not change. An easy way to see this is to follow the steps bellow to

construct YDs in the decomposition of the tensor product  $\mathcal{O} \otimes \psi$ , where  $\mathcal{O}$  is some YD in the same ensemble of  $\mathcal{O}_0$ .

- The effect of having sets of rows with equal length has at worst the effect of reducing  $N$  inside the log to a fraction of it, which does not modify the leading order (see the end of the previous section). A way to think about this is to imagine that we replaced our background Young tableau  $\mathcal{O}$  with a new one  $\overline{\mathcal{O}}$  by replacing all the sets of rows of equal length by a single row. By doing so we reduce the range of possible labels to just a fraction of  $N$ . But since we are not altering the inequalities  $h_i \ll n$ ,  $d$ , we can still use our original arguments used in the case of  $\mathcal{O}_0$ .
- After deflating  $\mathcal{O}$  to  $\overline{\mathcal{O}}$ , we first construct all the good labelings associated to the latter YD. After that, we inflate back  $\overline{\mathcal{O}}$  to the original YD  $\mathcal{O}$ . At the end of this operation we will end up with diagrams that are not YDs. Essentially, the rows right after each set of equal length rows of  $\mathcal{O}$  might be bigger than the one above it. But we can redistribute the extra boxes on the empty rows since they come from different rows in  $\psi$ , and hence they do not violate the antisymmetry rule. The ordering rule is also preserved because of the way we construct the tensor product if we move the boxes in the following way. We just cut the excess boxes and slide them up without changing their order. This is explained in the following example:



As a result, we do not change the leading order given in equation (F.4).

The second difference, the difference in length of consecutive rows is bigger than one box, is beneficial as it allows to have more options because now we can allow for more than one box with the same label  $i > n$ , for some  $i$ 's. But since already the leading behavior of the degeneracy is saturated by (F.4), we conclude once again that the leading term remains intact.

All in all, dealing with a different background YD in the same ensemble of  $\mathcal{O}_0$  leads to the same leading term of the largest degeneracy of YDs in the decomposition of  $\mathcal{O} \otimes \psi$  as in equation (F.4).

In the case of other ensembles, the same story goes through as the only change in this case can be reabsorbed by replacing  $N$  by a fraction of it in the worst case. This is because the background YDs we are dealing with are the ones where the number of rows is  $N$  and the number of columns  $N_c$  is of order  $N$ , and the number of boxes is fixed to be  $(N_c N_r)/2$ .

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